DIRECT, INVERSE AND SATURATION THEOREMS IN ORDINARY AND SIMULTANEOUS APPROXIMATION BY ITERATIVE COMBINATIONS OF EXPONENTIAL TYPE OPERATORS

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CERTIFICATE

Certified that the work presented in this thesis entitled 'DIRECT, INVERSE AND SATURATION THEOREMS IN ORDINARY AND SIMULTANEOUS APPROXIMATION BY ITERATIVE COMBINATIONS OF EXPONENTIAL TYPE OPERATORS' by Sri Purshottam Narain Agrawal has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

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CONTENTS

INTRODUCTION AND CONTENTS OF THE THESIS

For a given class $\{L_\lambda\}$ of approximation methods the investigation of its direct, inverse and saturation theorems forms one of the most-important theoretical and practical aspects of its study. A direct theorem provides the order of approximation for functions of a specified smoothness. On the other hand, an inverse theorem infers the nature of smoothness of a function from its order of approximation. A saturation theorem is a more curious phenomena. It refers to an inherent limitation (if present) of the approximation method : the order of approximation beyond a certain limit $O(\varphi(\lambda))$ $(\varphi(\lambda) \to o, \lambda \to \infty)$ is possible only for a trivial (finite dimensional) subspace. The functions for which the $O(\varphi(\lambda))$ approximation is attained form the saturation (Favard) class and those with the approximation order $o(\varphi(\lambda))$ come in the trivial class. Thus a saturation theorem consists of a determination of the saturation order $\varphi(\lambda)$, the saturation class and the trivial class. In this formulation the error in the approximation could be with respect to a certain function norm, seminorm or a metric, pointwise or local. The first case is often referred to as global. In the pointwise case the error term incorporates a function of the point under consideration and indicates a bias of the method towards certain points(usually the end

points of the interval). The common characteristic of global and pointwise results is their involvement with the whole domain(the set of definition) of functions, while the local results concern, both in hypothesis and conclusion, with limited subsets of the domain.

The study of direct theorems in approximation theory was initiated by the classical work of Jackson [18] on algebraic and trigonometric polynomials of best approximation. The corresponding inverse theorems were obtained by Bernstein(see e.g., Natanson [29]) by an ingeneous application of his famous inequality. In the trigonometric case the results of Jackson-Bernstein had an essential gap for the case $\alpha = 1$. This was filled, much later, by Zygmund [46] through the introduction of the class Z ($Lip^* 1$). Further generalizations of the Zygmund class have been found to be very useful in recent developments in approximation theory.

The Jackson-Bernstein theorems in the algebraic case were further improved by Timan and others(see Timan [44 and Lorentz [25]) by taking into account the improved natu of approximation near the end points. Alexits [1] initiate the study of saturation of convolution operators by characte the saturation class for Fejèr operators. A general formula of the phenomenon of saturation was given by Favard [13] an the saturation class bears his name. Zamansky [45] studied saturation of certain convolutions with trigonometric

polynomials in connection with summation methods applied to Fourier series. The Fourier coefficient method for trigonometric case of saturation was given by Sunouchi and Watari [39-40] (see also Buchwalter [6] and Harsilad For a detailed account of further earlier works we refer t [44] , [24] and [29].

The current interest in the above problems was reviv through the original work of Korovkin (see [21]) on linear positive operators. This resulted in a systematization of the earlier work on Bernstein polynomials, Fejér operators, Jackson and Weierstrass integrals etc. which, in retrospect turned out to be positive. The recognition of the basic role of linear positive operators triggered a virtual chain reaction in approximation theory. For a discussion of some of the more recent aspects we refer to De Vore [11]. Impor contributions to direct theorems are due to Freud [14-15], Sunouchi [38], Shapiro [36] (in whose work on convolutions with dilations of a kernel, direct and inverse theorems merg into a unified theme of comparison theorems) etc. For a systematic study of error estimates and Voronovskaja type asymptotic formulae (whose rôle in saturation theory is fundamental) we refer to Rathore [32] and the references therein.

The first result on saturation in the non-trigonometr case of Bernstein polynomials was obtained by de Leeuw [22]

His result was sharpened through a new functional analytic technique by Lorentz [24] (see also [25]). Lorentz and Schumaker [26] gave the first approach to pointwise saturati theorems (see also Berens [4]). Important contributions to saturation in local approximation were made by Saunouchi [37 (who for Fejér means used the asymptotic formulae involving the conjugate function derivative), Suzuki [41-42], Suzuki and Watanabe [43] , Ikeno and Suzuki [17] and others. Many of these and other works were based on the approach of Lorent Some recent works on saturation are due to De Vore [10-11], Schnabl [35], Becker, Kucharski and Nessel [3] etc.

Direct, inverse and saturation theorems in approximati by semigroups of operators had been extensively developed by and Berens [8]. Kelisky and Rivlin [20] and Karlin and Zieg [19] studied the convergence of iterates of certain sequenc of linear positive operators to semigroups of operators. Micchelli [28] based a theory of saturation of the original operators on this fact which provides an alternative to the mollifiers method (see Shapiro[36]) for operators which do not posses non-trivial commuting approximation operators. By this method Micchelli gave a unified approach for Lorentz type results for several sequences of linear positive operato

For operators possessing certain Bernstein-type inequal an approach towards inverse theorems is given in De Vore [11] Much work in this direction has been done by Butzer and his

associates. The study of inverse theorems for non-trigonometric and non-convolution type operators turns out to be much more difficult. In this connection the pioneering work of Berens and Lorentz [5] gives an elementry proof of global inverse theorem for Bernstein polynomials for the case $o < \alpha < 1$. An extension of their technique for the case $o < \alpha \leq 2$ has been made by Becker and Nessel [2]. For the interpolation theory approach to inverse results see Butzer and Scherer [9] (also Butzer and Berens [8]).

Several investigations (in our context starting with the work of Butzer [7] on Bernstein polynomials) indicated that even when a sequence or class $\{ L_\lambda \}$ of linear positive operators is saturated with a certain order of approximation, some carefully chosen linear combinations of its members may give a better order of approximation for smoother functions. A general direct theory of such combinations was furnished by Rathore [30] who based it on a certain notion of local unsaturation on positive zero orders. Micchelli [28] offered yet another approach for improving the order of approximation of Bernstein polynomials B_n by considering the iterative combinations $T_{n,k} = I - (I - B_n)^k$. What is interesting is that such combinations themselves turned out to be saturated with a certain (higher) order of approximation. The direct, inverse and saturation theorems for these combinations turn out to be much more complicated to obtain than the corresponding results for the original operators. Saturation of $T_{n,k}$ was established

by Micchelli [28] himself by using the semigroup approach. Only recently Ditzian and May [12] succeeded in obtaining local saturation theorems for Butzer type combinations of Bernstein polynomials and Szász operators (see also May [27] for an outline of the proof). The paper [27] of May introduces a class of the so called exponential type operators whose iterates are the subject of study of this thesis. This class among others contains the Bernstein polynomials, Szász - Mirakyan - Hille operators, a case of Baskakov operators, the Weierstrass integrals and an operator of Rathore [30]. In a unified treatment of this class May obtained inverse theorems for certain Butzer type linear combinations of members of this class. With an additional assumption of regularity he also obtained saturation theorems for these linear combinations. The treatment is based on intermediate space approach involving a Peetre's K-functional for inverse theorems and distribution theory oriented for saturation theorems.

Another topic of interest in the present thesis is the phenomena of simultaneous approximation (approximation of derivatives of functions by the corresponding order derivatives of operators). The first remarkable result in this direction is due to Lorentz (see [23]) who proved that $B_n^{(k)}(x) \to f^{(k)}(x)$, $n \to \infty$, whenever the latter exists at the particular point $x \in [0,1]$, $k = 1,2,3,\ldots$, being arbitrary. His method for this pointwise convergence in simultaneous approximation has since been applied by several workers to other operators.

Rathore [33-34] made a more detailed study of simultaneous approximation and established the existence of Voronovskaja type asymptotic formulae in simultaneous approximation. His results at once suggest the phenomena of saturation in simultaneous approximation and hence also the questions on direct and inverse theorems.

The main problem of the present thesis is the investigation of this possibility of direct, inverse and saturation theorems in simultaneous approximation for exponential type operators of May. In fact we have considered the problem in a much more general framework of the following two types of iterative combinations

$$S_{\lambda,k,m}(f,t) = \sum_{r=1}^{k} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m}{k-r} S_{\lambda}^{r+m}(f,t)$$

$$(k = 1,2,\ldots; \; m = 0,1,2,\ldots)$$

and

$$S_{\lambda,p}(f,k,t) = \frac{1}{\Delta} \begin{vmatrix} S^{p}_{d_0\lambda}(f,t) & d_0^{-1} & \cdots & d_0^{-k} \\ S^{p}_{d_1\lambda}(f,t) & d_1^{-1} & \cdots & d_1^{-k} \\ \vdots & \vdots & & \vdots \\ S^{p}_{d_k\lambda}(f,t) & d_k^{-1} & \cdots & d_k^{-k} \end{vmatrix}$$

($k = 0,1,\ldots$; $p = 1,2,3,\ldots$ and d_j's are distinct positive real no's),

where Δ is the determinant obtained by replacing the operator column in the above determinant by the entries 1. Our results contain direct, inverse and saturation theorems in both the ordinary and simultaneous approximation.

It is remarked that the iterative combinations of Micchelli as well as the ordinary combinations considered by May all turn out to be very special cases of the above two combinations. Also, as will be seen later our approach to the local direct, inverse and saturation theorems is considerably simpler and unified than the other approaches mentioned earlier.

The thesis consists of five chapters whose contents are as described below :

Chapter 1 contains the formal definition of exponential type operators and the constraints of the notion of regularity. Our definitions are appropriately modified than those in May [27] The results of May still hold and are described next. These are also supplemented by a result on direct error estimate. We then introduce a notion of certain dual operators corresponding to the regular case of exponential type operators. This notion results in a considerable simplification of the saturation theory. Some approximation results for these dual operators are also proved. In the last section as a motivation we state the direct, inverse and saturation results in simultaneous approximation by

$S_{\lambda,1}(f,k,t)$ which are to be proved in Chapter 4 for the general case $S_{\lambda,p}(f,k,t)$.

Chapter 2 formally introduces the combinations $S_{\lambda,k,m}$ and is a study of ordinary approximation by these operators. Basic results on the moments and asymptotic formulae for these iterative combinations are derived first. Then we develop the direct and inverse theorems for these combinations of general exponential type operators and the saturation theorem for regular exponential type operators.

Chapter 3 is a study of the problems of Chapter 2 in simultaneous approximation. Here we introduce a new Peetre's functional $K_p(\xi,f)$ and define an intermediate space $C_0^p(\alpha,k;a',b')$ with its help. This along with an equivalence lemma provides the basic machinery for inverse theorems. For tackling the saturation problem a very interesting approach is made in which a 'switching of the derivatives' (see equation (3.4.2)) enables the ordinary approximation methods to be applicable also in the simultaneous approximation case.

Chapter 4 is a combined study of direct, inverse and saturation problems for the combinations $S_{\lambda,p}(f,k,t)$ both in the ordinary and the simultaneous approximation cases.

Chapter 5 considers and completes the local saturation theory for the two combinations $S_{\lambda,k,m}(f,t)$ and $S_{\lambda,p}(f,k,t)$ of Bernstein polynomials both in the ordinary and the

simultaneous approximation. Here the approach had to be different for the Bernstein polynomials do not satisfy the regularity assumption and therefore our technique of the dual operators becomes inapplicable. The results show that these combinations of the Bernstein polynomials also have the same saturation behaviour as that of the corresponding combinations of the regular exponential type operators.

We would like to mention that the study of iterates of operators is not just academic. One of the practical reasons for their investigation is the enhancement of several qualitative approximation properties (e.g., the variation diminishing property) in the iterates.

CHAPTER 1

SOME BASIC RESULTS ON EXPONENTIAL TYPE OPERATORS

1.1 INTRODUCTION

In this chapter we introduce the exponential type operators S_λ and their linear combinations $S_\lambda(f,k,t)$ in a slightly modified form than that considered by May [27] and present some basic results on pointwise simultaneous approximation of derivatives of functions by the derivatives of the exponential type operators. We also summarize the results of May which remain valid for the operators under consideration and include some results on the dual operators S^*_λ in the case when the operators S_λ are regular.

In later chapters we shall study two types of iterative combinations of the operators S_λ and there the results of this chapter will be of an extensive use.

1.2 EXPONENTIAL TYPE OPERATORS

Let Λ denote an unbounded subset of $\mathbb{R}^+$ (the set of all positive numbers). A family $\{S_\lambda, \lambda \in \Lambda\}$ of operators having the form

(1.2.1) $S_\lambda(f,t) = \int_A^B W(\lambda,t,u)\, f(u)\, du,$

where $-\infty \leq A < B \leq \infty$ and $W(\lambda,t,u) \geq o$ is the distributional kernel, is said to be of exponential type if there hold

(1.2.2) $\int_A^B W(\lambda,t,u)\, du = 1,\ t \in (A,B)$, and

(1.2.3) $\frac{\partial}{\partial t} W(\lambda,t,u) = \frac{\lambda}{p(t)} W(\lambda,t,u)\,(u-t),\ u,t \in (A,B),$

where $p(t)$ is a positive second degree polynomial on (A,B).

It is further assumed that the range of S_λ is contained in $C^\infty(A,B)$ (the class of infinitely differentiable functions on the interval (A,B)) and moreover that there holds

$$(1.2.4) \quad \frac{d^k}{dt^k} S_\lambda(f,t) = \int_A^B \frac{\partial^k}{\partial t^k} [\, W(\lambda,t,u)\,] f(u)\, du,$$

for $k \in \mathbb{N}$.

The nature of the domain of the operators S_λ varies with the nature of $W(\lambda,t,u)$. Thus, for instance, for the operators of Post-Widder and Gauss-Weierstrass, defined respectively by

$$S_\lambda^1(f,t) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{t}\right)^\lambda \int_0^\infty e^{-\lambda u/t}\, u^{\lambda-1} f(u)\, du \quad (\lambda \in \mathbb{R}^+),$$

and

$$S_\lambda^2(f,t) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_{-\infty}^\infty e^{-\lambda(u-t)^2/2} f(u)\, du \quad (\lambda \in \mathbb{R}^+),$$

f must be measurable on $\mathbb{R}^+$ and $\mathbb{R} = (-\infty,\infty)$; while for the Bernstein polynomials, the Szász operators and the Baskakov operators, defined respectively by

$$S_\lambda^3(f,t) = \sum_{k=0}^{\lambda} \binom{\lambda}{k} t^k (1-t)^{\lambda-k} f\left(\frac{k}{\lambda}\right) \quad (\lambda \in \mathbb{N}),$$

$$S_\lambda^4(f,t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda t)^k f\left(\frac{k}{\lambda}\right) \quad (\lambda \in \mathbb{R}^+) \quad \text{and}$$

$$S_\lambda^5(f,t) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi_\lambda^{(k)}(t)}{k!} t^k f\left(\frac{k}{\lambda}\right) \quad (\lambda \in \mathbb{R}^+),$$

where the family $\{\phi_\lambda\}$ satisfies certain appropriate conditions, the measurability of f is not required.

Thus, we do not specify the domain of the operators S_λ explicitly ; we only require that the distributional integrals in (1.2.1) and (1.2.4) remain meaningful. In the sequel writing $S_\lambda(f,t)$ entails the assumption that it be defined.

The operators S^1_λ to S^5_λ as shown by May [27] are of exponential type with the kernels respectively given by

$$W_1(\lambda, t, u) = \frac{1}{\Gamma(\lambda)} \left(\frac{\lambda}{t}\right)^\lambda e^{-\lambda u/t} \; u^{\lambda-1} ,$$

$$W_2(\lambda, t, u) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \; e^{-\lambda(u-t)^2/2} ,$$

$$W_3(\lambda, t, u) = \sum_{k=o}^{\lambda} \binom{\lambda}{k} t^k (1-t)^{\lambda-k} \; \delta(u-\tfrac{k}{\lambda}),$$

$$W_4(\lambda, t, u) = e^{-\lambda t} \sum_{k=o}^{\infty} \frac{1}{k!} (\lambda t)^k \delta(u-\tfrac{k}{\lambda}) \quad \text{and}$$

$$W_5(\lambda, t, u) = \sum_{k=o}^{\infty} (-1)^k \; \frac{\phi_\lambda^{(k)}(t)}{k!} \; t^k \delta(u-\tfrac{k}{\lambda}),$$

where $\delta(x)$ denotes the Dirac δ-function.

Also, the operators L_k of Rathore [30, p.157] defined by

$$L_k(f,u) = (1+u)^{-(k+1)} \sum_{s=o}^{\infty} \binom{k+s}{s} \left(\frac{u}{1+u}\right)^s f\left(\frac{s}{k+p}\right)$$

give rise to the operators

$$S^6_\lambda(f,t) = (1+t)^{-\lambda} \sum_{s=o}^{\infty} \binom{\lambda+s-1}{s} \left(\frac{t}{1+t}\right)^s f\left(\frac{s}{\lambda}\right) \; (\lambda \in \mathrm{IN}),$$

where $S^6_\lambda \equiv L_{\lambda-1}$ with $p = 1$, which are of exponential type with the kernel

$$W_6(\lambda, t, u) = (1+t)^{-\lambda} \sum_{s=o}^{\infty} \binom{\lambda+s-1}{s} \left(\frac{t}{1+t}\right)^s \; \delta(u-\tfrac{s}{\lambda}).$$

The operators S_λ of exponential type are called <u>regular</u> if $W(\lambda, t, u)$ are measurable functions of (t, u) and there holds

(1.2.5) $\int_A^B W(\lambda, t, u)\, dt = a(\lambda), \; u \in (A, B),$

where $a(\lambda)$ is a rational function of λ satisfying

(1.2.6) $\lim\limits_{\lambda \to \infty} a(\lambda) = 1,$

the limit being taken along $\lambda \in \Lambda$ and further that

(1.2.7) for each fixed $u \in (A,B)$ and $m \in \mathbb{N}^o$ (the set of non-negative integers), $t^m\, p(t)\, W(\lambda, t, u) \to o$ as $t \to A, B$, for all λ sufficiently large.

Let $d_o, d_1, \ldots, d_k$ ($k \in \mathbb{N}^o$, the set of non-negative integers) be k+1 arbitrary but fixed distinct positive real numbers. Then following Rathore [30] for $\lambda \in \Lambda$ such that $d_i\lambda \in \Lambda$, $i = o, 1, \ldots, k$ the <u>linear combination</u> $S_\lambda(f, k, t)$ of $S_{d_j\lambda}(f, t)$, $j = o, 1, \ldots, k$ is defined by

(1.2.8) $$S_\lambda(f,k,t) = \frac{1}{\Delta} \begin{vmatrix} S_{d_o\lambda}(f,t) & d_o^{-1} & d_o^{-2} & \ldots & d_o^{-k} \\ S_{d_1\lambda}(f,t) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \vdots & \vdots & \vdots & & \vdots \\ S_{d_k\lambda}(f,t) & d_k^{-1} & d_k^{-2} & & d_k^{-k} \end{vmatrix}.$$

where Δ is the Vandermonde determinant

$$(1.2.9)\qquad \Delta = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix} .$$

On a simplification, we have

$$(1.2.10)\quad S_\lambda(f,k,t) = \sum_{j=o}^{k} C(j,k)\, S_{d_j\lambda}(f,t), \text{ where}$$

$$(1.2.11)\qquad C(j,k) = \prod_{\substack{i=o \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \; k \neq o \text{ and } C(o,o) = 1.$$

We close this section with a few remarks : (i) May [27] does not mention the property (1.2.7) explicitly even though he uses it in his paper (in the proof of Lemma 8.1, while deriving (8.6)) ; (ii) he also utilizes (1.2.4) (in the proof of Lemma 4.2) without assuming it explicitly and (iii) in his combinations $S_\lambda(f,k,t)$, May assumes the numbers d_j's to be integers. However, it is easily verified that his results remain true for our more general combinations.

1.3 PRELIMINARIES AND SUPPLEMENTARY RESULTS

The k-th order modulus of continuity $\omega_k(f;\delta)$ for a function f continuous on an interval I is defined by

$$\omega_k(f;\delta) = \sup \{ \; |\Delta_h^k f(x)| \; : |h| \leq \delta, x, x+kh \in I \}.$$

The function f is said to belong to the generalized Zygmund class Liz $(\alpha, k; a, b)$ if there exists a constant M such that

$$\omega_{2k}(f;\delta) \leq M\,\delta^{\alpha k}, \quad \delta > 0,$$

where $\omega_{2k}(f;\delta)$ denotes the modulus of continuity of 2k-th order on the interval $[a,b]$. The class Liz $(\alpha, 1; a, b)$ is more commonly denoted by $\mathrm{Lip}^{*}(\alpha; a, b)$ or $Z_{\alpha}(a,b)$.

Let C_0 denote the set of continuous functions on (A,B) having a compact support and C_0^k the subset of C_0 of k-times continuously differentiable functions. With $[a,b] \subset (A,B)$ and $G(a,b) := \{g \in C_0^{2k+2}, \operatorname{supp} g \subset [a,b]\}$, the K-functional $K(\xi, f; a, b)$ $(0 < \xi \leq 1)$ for $f \in C_0$ with $\operatorname{supp} f \subset [a,b]$ is defined by

$$K(\xi, f; a,b) = \inf_{g \in G(a,b)} \{\|f-g\| + \xi(\|g\| + \|g^{(2k+2)}\|)\},$$

where $\|.\|$ denotes the sup-norm on (A,B). For $0<\alpha<2$, $C_0(\alpha, k+1; a, b)$ denotes the set of functions f for which

$$\|f\|_{\alpha} \equiv \sup_{0<\xi\leq 1} \xi^{-\alpha/2} K(\xi, f; a,b) < M,$$

for some constant M.

In the sequel $\lambda \in \Lambda$ and S_λ stands for an operator of exponential type from the set $\{S_\lambda, \lambda \in \Lambda\}$ with $W(\lambda, t, u)$ denoting the kernel of S_λ. May [27] proved the following results :

LEMMA 1.3.1 : If f is a polynomial of degree $\leq k$, then so is $S_\lambda(f,t)$.

LEMMA 1.3.2 : If $\delta>o$, $m>o$ and $A<a<b<B$, then

(1.3.1) $$\int_{|u-t|\geq\delta} W(\lambda, t, u)\, du = O(\lambda^{-m}),\ \lambda \to \infty,$$

uniformly for $t \in [a,b]$.

LEMMA 1.3.3 : If

(1.3.2) $$A_m(\lambda, t) = \lambda^m \int_A^B W(\lambda, t, u)(u-t)^m\, du,\ m \in \mathbb{N}^o,$$

then $A_o(\lambda, t) = 1$, $A_1(\lambda, t) = o$ and

(1.3.3) $$A_{m+1}(\lambda, t) = \lambda m\, p(t)\, A_{m-1}(\lambda, t) + p(t)\, A_m'(\lambda, t),\ m \in \mathbb{N}$$ (the set of natural numbers).

Consequently, $A_m(\lambda, t)$ is a polynomial in t and λ of a degree $\leq m$ in t and of degree $[\frac{m}{2}]$ in λ, $[x]$ denoting the integral part of x. The coefficient of λ^m in the polynomial $A_{2m}(\lambda, t)$ is $(2m-1)!!\, p^m(t)$, $a!!$ denoting the semifactorial of a. Also, the coefficient of λ^m in $A_{2m+1}(\lambda, t)$ is $C_m p^m(t)p'(t)$, where C_m is a constant.

An explicit evaluation of C_m was not made by May, which, however, is given by

(1.3.4) $$C_m = \frac{(2m+1)!!\, m}{3},$$

and can be obtained as follows : Assuming (1.3.4) for a certain m, by (1.3.2) we have

$$A_{2m+3}(\lambda, t) = 2\lambda(m+1)p(t)A_{2m+1}(\lambda, t) + p(t)A_{2m+2}'(\lambda, t).$$

Hence the coefficient of λ^{m+1} in $A_{2m+3}(\lambda,t)$ is

$$2(m+1)p(t)(2m+1)!!\,\frac{m}{3}\,p^m(t)p'(t)+p(t)(2m+1)!!(p^{m+1}(t))'$$

$$=\{2(m+1)(2m+1)!!\,\frac{m}{3}+(2m+1)!!(m+1)\}\,p^{m+1}(t)p'(t)$$

$$=\frac{(2m+3)!!(m+1)}{3}p^{m+1}(t)p'(t)$$

showing that (1.3.4) also holds for $m+1$.

Since $A_1(\lambda,t)=o$, (1.3.4) holds for $m=o$. Hence by induction (1.3.4) follows for all $m \in \mathbb{N}^o$.

A function $\Psi(\geq 1) \in C(A,B)$ will be called a <u>growth test function</u> (GTF) for $\{S_\lambda, \lambda \in \Lambda\}$ if for any compact subset K of (A,B) there exists a $\lambda_o \in \Lambda$ and a positive constant M such that

(1.3.5) $\quad S_\lambda(\Psi^2,t) < M, \ \lambda > \lambda_o, \ t \in K.$

The function $(1+t^2)^N$, $N > o$ is always a GTF for exponential type operators. As mentioned by May [27], he and Ismail have shown that $e^{N|t|}$, $N > o$ is also a GTF for the operators S_λ. Throughout this chapter Ψ denotes a GTF for S_λ.

The set of all functions f which belong to the domain of S_λ for all λ sufficiently large and satisfy $|f(t)| \leq M\Psi(t)$, $t \in (A,B)$, for some constant M is denoted by $D_\Psi(A,B)$. $C_\Psi(A,B)$ stands for the subset of $D_\Psi(A,B)$ consisting of continuous functions. The space $C_\Psi(A,B)$ is normed by

(1.3.6) $\quad ||f||_{C_\Psi} = \sup_{t \in (A,B)} \{|f(t)|/\Psi(t)\}.$

Let $\langle a,b\rangle \subset (A,B)$ denote an open interval containing the closed interval $[a,b]$.

THEOREM 1.3.4 : If $f \in D_\psi(A,B)$ and is continuous at a point $t \in (A,B)$, then

$$(1.3.7) \qquad \lim_{\lambda \to \infty} S_\lambda(f,k,t) = f(t).$$

Also, if f is continuous on $\langle a,b\rangle$, then (1.3.7) holds uniformly for $t \in [a,b]$.

THEOREM 1.3.5 : If $f \in D_\psi(A,B)$ and $f^{(2k+2)}$ exists at some point $t \in (A,B)$, then

$$(1.3.8) \qquad \lim_{\lambda \to \infty} \lambda^{k+1} [\, S_\lambda(f,k,t) - f(t)\,] = \sum_{j=k+1}^{2k+2} Q(j,k,t) f^{(j)}(t),$$

where $Q(j,k,t)$ are certain polynomials in t. Moreover,

$Q(2k+2,k,t) = C_1 p^{k+1}(t)$ and $Q(2k+1,k,t) = C_2 p^k(t) p'(t)$, C_1 and C_2 being certain constants.

Also, if $f^{(2k+2)}$ exists and is continuous on $\langle a,b\rangle$, then (1.3.8) holds uniformly for $t \in [a,b]$.

For $k = o$ and $d_o = 1$, (1.3.8) reduces to the Voronovskaya formula

$$(1.3.9) \qquad \lim_{\lambda \to \infty} \lambda [\, S_\lambda(f,t) - f(t)\,] = \frac{1}{2} p(t) f''(t).$$

We also mention the following result from May [27].

THEOREM 1.3.6 : If $f \in C_o$ with supp $f \subset (a,b)$ then $f \in Liz(\alpha,k+1;a,b)$ if, and only if $f \in C_o(\alpha,k+1;a,b)$.

The inverse and saturation theorems which May proved for the combinations $S_\lambda(f,k,t)$ are as follows : Let $f \in C_\psi(A,B)$ and $A < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < B$. Let $\{\lambda_n \in \Lambda\}$ be a monotonically divergent subsequence such that for some constant C there holds $\lambda_{n+1}/\lambda_n \le C$, $n \in \mathbb{N}$. Throughout this thesis we assume that a_i, b_i, $i = 1,2,3$ and λ_n, $n \in \mathbb{N}$ satisfy the conditions of this paragraph.

<u>THEOREM</u> 1.3.7 : If $0 < \alpha < 2$, in the following statements the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) hold .

(i) $\quad ||S_{\lambda_n}(f,k,t) - f(t)||_{C[a_1,b_1]} = O(\lambda_n^{-\alpha(k+1)/2})$;

(ii) $\quad f \in \mathrm{Liz}(\alpha,k+1;a_2,b_2)$;

(iii) (a) If $m < \alpha(k+1) < m+1$, $m = 0,1,2,\ldots,2k+1$, then $f^{(m)}$ exists and $\in \mathrm{Lip}(\alpha(k+1) - m\,;\,a_2,b_2)$,

(b) If $\alpha(k+1) = m+1$, $m = 0,1,2,\ldots,2k$, then $f^{(m)}$ exists and $\in \mathrm{Lip}^*(1;a_2,b_2)$;

(iv) $\quad ||S_\lambda(f,k,t) - f(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2})$.

<u>THEOREM</u> 1.3.8 : If S_λ are regular and $I(f,\lambda,k,a,b) = \lambda^{k+1}||S_\lambda(f,k,t) - f(t)||_{C[a,b]}$, in the following statements, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) hold.

(i) $I(f, \lambda_n, k, a_1, b_1) = O(1)$;

(ii) $f^{(2k+1)} \in A.C.\ [a_2, b_2]$ and $f^{(2k+2)} \in L_\infty [a_2, b_2]$;

(iii) $I(f, \lambda, k, a_3, b_3) = O(1)$;

(iv) $I(f, \lambda_n, k, a_1, b_1) = o(1)$;

(v) $f \in C^{2k+2} [a_2, b_2]$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t) f^{(i)}(t) = o,\ t \in [a_2, b_2]$

where $Q(i,k,t)$ are the polynomials occuring in (1.3.8);

(vi) $I(f, \lambda, k, a_3, b_3) = o(1)$.

We supplement the above results by proving a direct error estimate involving a local modulus of continuity of derivatives of f.

<u>THEOREM</u> 1.3.9 : Let $o \leq p \leq 2k+2$, $f \in D_\psi(A,B)$ and $f^{(p)}$ exist and be continous on $\langle a,b\rangle$. Then, for all λ sufficiently large, there holds

$$(1.3.10)\quad \| S_\lambda (f,k,t) - f(t) \|_{C[a,b]} \leq \max \{ C\lambda^{-p/2}\, \omega(f^{(p)};\ \lambda^{-1/2}),\ C'\lambda^{-(k+1)} \},$$

where $C = C(k,p)$, $C' = C'(k,p,f)$ and $\omega(f^{(p)};\delta)$ denotes the modulus of continuity of $f^{(p)}$ on $\langle a,b\rangle$.

<u>PROOF</u> : For $t \in [a,b]$, writing

$$(1.3.11)\quad F(u,t) = f(u) - f(t) - \sum_{j=1}^{p} \frac{f^{(j)}(t)}{j!} (u-t)^j,$$

if $u \in \langle a,b \rangle$, we have

$$(1.3.12) \quad |F(u,t)| \leq \frac{|u-t|^p}{p!}\left(1 + \frac{|u-t|}{\lambda^{-1/2}}\right) \omega(f^{(p)} ; \lambda^{-1/2}).$$

Thus, if χ denotes the characteristic function of $\langle a,b \rangle$, by Lemma 1.3.3 and Schwarz inequality there follows

$$(1.3.13) \quad S_\lambda(|F(u,t)|\, \chi(u),t) \leq C_1 \lambda^{-p/2} \omega(f^{(p)}; \lambda^{-1/2}),$$

where $C_1 = C_1(p)$. Similarly, for some constant C_2 and all λ sufficiently large, we have

$$S_\lambda(|F(u,t)|\ (1 - \chi(u)),t)$$

$$\leq C_2 \left[S_\lambda((u-t)^{2(2k+2)},t)\ S_\lambda(\Psi^2,t) \right]^{1/2}$$

$$\leq C_3 \lambda^{-(k+1)} .$$

Hence $|S_\lambda(F(u,t),t)| \leq C_1 \lambda^{-p/2} \omega(f^{(p)} ; \lambda^{-1/2}) + C_3 \lambda^{-(k+1)}$.

But, by Theorem 1.3.5

$$S_\lambda\left(\sum_{j=1}^{p} \frac{f^{(j)}(t)}{j!} (u-t)^j,k,t\right) = O(\lambda^{-(k+1)}),$$

uniformly in $t \in [a,b]$. Hence for all λ sufficiently large we have

$$\| S_\lambda(f,k,t)-f(t) \|_{C[a,b]} \leq C_4 \lambda^{-p/2} \omega(f^{(p)}; \lambda^{-1/2}) + C_5 \lambda^{-(k+1)}$$

where C_4 does not depend on f, from which (1.3.10) is immediate.

This completes the proof.

1.4 THE DUAL OPERATORS

Let $\{S_\lambda\}$ be a regular family of exponential type operators. Then the dual operator S_λ^* corresponding to S_λ is defined by

(1.4.1) $$S_\lambda^*(f(t),u) = \int_A^B W(\lambda,t,u)\, f(t)dt .$$

The domain of S_λ^* is the set of functions f for which the right hand side of (1.4.1) is meaningful. Let $\langle f,g\rangle$ denote the inner product $\int_A^B f(t)g(t)dt$. The reason for calling S_λ^* to be the dual of S_λ is as follows : If $S_\lambda f$, $S_\lambda^* g$, $\langle S_\lambda f,g\rangle$ and $\langle f,S_\lambda^* g\rangle$ are defined, we have the adjoint relation
$\langle S_\lambda f,g\rangle = \langle f,S_\lambda^* g\rangle$.

For, we have

$$\langle f(u),\ S_\lambda^*(g(t),u)\rangle$$

$$= \int_A^B f(u)\left(\int_A^B W(\lambda,t,u)\, g(t)dt\right) du$$

$$= \int_A^B g(t)\left(\int_A^B W(\lambda,t,u)\, f(u)\, du\right) dt$$

$$= \langle S_\lambda(f(u),t),\ g(t)\rangle ,$$

by Fubini's theorem.

The k-th moment $\mu_{\lambda,k}^*(u)$ of S_λ^* is defined, as usual, by

(1.4.2) $$\mu_{\lambda,k}^*(u) = S_\lambda^*((t-u)^k,u),\ k \in \mathbb{N}^0.$$

The normalized k-th moment $\sigma_{\lambda,k}^*(u)$ is defined by

(1.4.3) $$\sigma_{\lambda,k}^*(u) = [a(\lambda)]^{-1}\ \mu_{\lambda,k}^*(u),\ k \in \mathbb{N}^0.$$

Clearly $\sigma^*_{\lambda,o}(u) = 1$.

A recursion relation for the normalized moments is as follows :

LEMMA 1.4.1 : For each $k \in \mathbb{N}^o$, there holds

$$(1.4.4)\qquad \{\lambda - (k+2)\alpha\}\sigma^*_{\lambda,k+1}(u) = 2(k+1)(\alpha u+\beta)\,\sigma^*_{\lambda,k}(u)+kp(u)\,\sigma^*_{\lambda,k-1}(u)$$

for all λ sufficiently large,

where $p(t) = \alpha t^2+2\beta t+\gamma$ is as in (1.2.3) and $\sigma^*_{\lambda,-1}(u) \equiv o$.

PROOF : By definition, for all λ sufficiently large we have

$$\sigma^*_{\lambda,k+1}(u) = [\,a(\lambda)\,]^{-1} \int_A^B W(\lambda,t,u)(t-u)^{k+1}\,dt$$

$$= -[\,a(\lambda)\,]^{-1} \int_A^B \left(\frac{p(t)}{\lambda}\,\frac{\partial}{\partial t} W(\lambda,t,u)\right)(t-u)^k\,dt$$

$$= [\,a(\lambda)\,]^{-1} \int_A^B \left(\frac{p(t)\,(t-u)^k}{\lambda}\right)'_t W(\lambda,t,u)\,dt,$$

using integration by parts, (1.2.3) and (1.2.7). Hence

$$\sigma^*_{\lambda,k+1}(u) = [\lambda\,a(\lambda)\,]^{-1} \int_A^B [\,\alpha(t-u)^{k+2} + 2\,(\alpha u+\beta)(t-u)^{k+1}$$

$$+\,p(u)(t-u)^k\,]'_t\,W(\lambda,t,u)\,dt$$

$$= [\lambda\,a(\lambda)\,]^{-1} \int_A^B [\,\alpha(k+2)(t-u)^{k+1} + 2(k+1)(\alpha u+\beta)(t-u)^k$$

$$+kp(u)(t-u)^{k-1}\,]\;W(\lambda,t,u)\,dt,$$

where for $k = o$ the last term within the square brackets is absent.

From this (1.4.4) is immediate.

COROLLARY 1.4.2 : For all λ sufficiently large, $\mu^*_{\lambda,k}(u)$ ($k \in \mathbb{N}^o$) is a polynomial in u of degree $\leq k$. Moreover

$$(1.4.5) \qquad \mu^*_{\lambda,k}(u) = O(\lambda^{-[\frac{k+1}{2}]}).$$

The proof of Corollary 1.4.2 follows from Lemma 1.4.1 by an induction on k. We also remark that in view of (1.2.5-7) each $\mu^*_{\lambda,k}(u)$ exists for all λ sufficiently large. ∎

The operators S^*_λ are themselves approximation methods. Let Φ be a GTF for the operators S^*_λ (i.e. $\Phi(\geq 1) \in C(A,B)$ and that for each compact $K \subset (A,B)$ there exist λ_o, M such that $S^*_\lambda(\Phi^2,u) < M$, $\lambda > \lambda_o$ and $u \in K$). The function $(1+t^2)^N$ is always a GTF for S^*_λ. The function spaces $D_\Phi(A,B)$ and $C_\Phi(A,B)$ are defined analogous to $D_\psi(A,B)$ and $C_\psi(A,B)$. Similarly the combinations $S^*_\lambda(f,k,u)$ are defined by (1.2.8) after replacing $S_{d_j\lambda}(f,t)$ by $S^*_{d_j\lambda}(f,u)$, $j = o,1,\dots,k$.

THEOREM 1.4.3 : If $f \in D_\Phi(A,B)$ and is continous at a point $u \in (A,B)$, then

$$(1.4.6) \qquad \lim_{\lambda\to\infty} S^*_\lambda(f,k,u) = f(u).$$

Also, (1.4.6) holds uniformly on $[a,b]$ if f is continuous on $\langle a,b\rangle$.

PROOF : To prove (1.4.6), obviously it is sufficient to show that

$$(1.4.7) \qquad \lim_{\lambda\to\infty} S^*_\lambda(f,u) = f(u)$$

and that it holds uniformly in the uniformity case. Since $f \in D_\phi$, if u belongs to a compact set K and χ_δ denotes the characteristic function of $[u-\delta, u+\delta]$, where δ is a fixed positive number, it is easily seen by (1.4.5) and the Schwarz inequality that

$$(1.4.8) \quad \lim_{\lambda \to \infty} S_\lambda^* (|f(t) - f(u)| \; (1-\chi_\delta(t)), u) = o$$

uniformly in $u \in K$.

But, due to the continuity of f at u, given an arbitrary $\epsilon > o$ we can choose the above $\delta > o$ such that

$$|f(t) - f(u)| \le \epsilon, \; |t-u| < \delta$$

and that in the uniformity case this δ can be chosen to be independent of $u \in [a,b]$. Thus

$$(1.4.9) \quad S_\lambda^* (|f(t) - f(u)| \, \chi_\delta(t), u) \le \epsilon \, a(\lambda).$$

Combining (1.4.8 - 9), by (1.2.6) we have

$$\overline{\lim_{\lambda \to \infty}} \, |S_\lambda^*(f(t) - f(u), u)| \le \epsilon,$$

from which, again using (1.2.6) and the arbitrariness of $\epsilon > o$, we have (1.4.7) and that it holds uniformly in $u \in [a,b]$ if f is continuous on $\langle a,b \rangle$.

This completes the proof.

<u>THEOREM</u> 1.4.4 : Let $k \in \mathbb{N}^o$, $f \in D_\phi(A,B)$ and $f^{(2k+2)}$ exist at a point $u \in (A,B)$. Then

$$(1.4.10) \; S_\lambda^*(f,u) - f(u) = (a(\lambda)-1)f(u) + \sum_{j=1}^{2k+2} \frac{f^{(j)}(u)}{j!} \mu_{\lambda,j}^*(u) + o(\lambda^{-(k+1)})$$

Further, if $f^{(2k+2)}$ exists and is continuous on $\langle a,b\rangle$ then (1.4.10) holds uniformly in $u \in [a,b]$.

The proof of this theorem follows from (1.4.5) along lines somewhat similar to that of the proof of Theorem 1.4.3 and is omitted.

An immediate consequence of Theorem 1.4.4 is the

<u>COROLLARY</u> 1.4.5 : If $k \in \mathbb{N}^o$, $f \in D_\varphi(A,B)$, $u \in (A,B)$ and $f^{(2k+2)}(u)$ **exists, then**

$$(1.4.11)\ \lim_{\lambda\to\infty} \lambda^{k+1}[\,S_\lambda^*(f,k,u) - f(u)\,] = \sum_{j=o}^{2k+2} \frac{f^{(j)}(u)}{j!}\, Q^*(j,k,u),$$

where $Q^*(j,k,u)$ is the coefficient of $\lambda^{-(k+1)}$ in the asymptotic expansion of $\mu^*_{\lambda,j}(u)$, multiplied by $(-1)^k/\prod_{i=o}^{k} d_i$. Moreover, (1.4.11) holds uniformly in $u \in [a,b]$ if $f^{(2k+2)}$ exists and is continuous on $\langle a,b\rangle$.

Also, along the lines of the proof of Theorem 1.3.9, we can easily prove

<u>THEOREM</u> 1.4.6 : Let $o \leq p \leq 2k+2$, $f \in D_\varphi(A,B)$, $f^{(p)}$ exist and be continuous on $\langle a,b\rangle$. Then, for all λ sufficiently large

$$(1.4.12)\ ||S_\lambda^*(f,k,u) - f(u)||_{C[a,b]} \leq \max\{\, C\lambda^{-p/2}\,\omega(f^{(p)};\lambda^{-1/2}),\ C'\lambda^{-(k+1)}\,\},$$

where $C = C(k,p)$, $C' = C'(k,p,f)$ and $\omega(f^{(p)};\delta)$ denotes the modulus of continuity of $f^{(p)}$ on $\langle a,b\rangle$.

Our next result gives a qualitative expansion of iterates of S_λ^* for functions $\in C_o^\infty(A,B)$, the space of infinitely differentiable functions of compact support in (A,B).

<u>THEOREM</u> 1.4.7 : Let $r,m \in \mathbb{N}$ and $f \in C_o^\infty(A,B)$. Then

$$(1.4.13) \quad S_\lambda^{*r}(f,u) = \sum_{i=o}^{r} \binom{r}{i} F_{i,m}(u,\lambda) + o(\lambda^{-m}),$$

uniformly on each compact subset of (A,B) where the functions $F_{i,m}$ are of the form

$$(1.4.14) \quad F_{i,m}(u,\lambda) = \sum_{j=1}^{m} \frac{f_{i,m,j}(u)}{\lambda^j}$$

for some functions $f_{i,m,j}(u) \in C_o^\infty(A,B)$ depending on f, $F_{o,m}(u,\lambda) = f(u)$ and

$$(1.4.15) \quad S_\lambda^*(F_{i,m}(t,\lambda),u) = F_{i,m}(u,\lambda) + F_{i+1,m}(u,\lambda) + o(\lambda^{-m}),$$

$$i = o,1,\ldots,r-1.$$

<u>PROOF</u> : For $r = 1$, the result follows from Theorem 1.4.4. For a general r we proceed by induction on r. Thus assume that (1.4.13) holds for a particular $r \in \mathbb{N}$. It is clear that the $o(\lambda^{-m})$ term in (1.4.13) is a bounded function of u on (A,B), since all the other terms in (1.4.13) determine bounded functions of u on (A,B). Thus operating (1.4.13) by S_λ^* we have

$$S_\lambda^{*r+1}(f,u) = \sum_{i=o}^{r} \binom{r}{i} S_\lambda^*(F_{i,m}(t,\lambda),u) + o(\lambda^{-m})$$

$$= \sum_{i=o}^{r-1} \binom{r}{i} [F_{i,m}(u,\lambda) + F_{i+1,m}(u,\lambda)] + S_\lambda^*(F_{r,m}(t,\lambda),u) + o(\lambda^{-m})$$

$$= \sum_{i=o}^{r} \binom{r+1}{i} F_{i,m}(u,\lambda) + [S_\lambda^*(F_{r,m}(t,\lambda),u) - F_{r,m}(u,\lambda)] + o(\lambda^{-m}).$$

But, using Theorem 1.4.4 we can write

$$S_\lambda^*(f_{r,m,j}(t),u) = f_{r,m,j}(u) + \sum_{\ell=1}^{m} \frac{f_{r,m,j,\ell}(u)}{\lambda^\ell} + o(\lambda^{-m}),$$

where $f_{r,m,j,\ell} \in C_0^\infty$ are certain functions depending on $f_{r,m,j}$.
Hence

$$S_\lambda^*(F_{r,m}(t,\lambda),u) = F_{r,m}(u,\lambda) + \sum_{j=r}^{m} \lambda^{-j} \sum_{\ell=1}^{m} \lambda^{-\ell} f_{r,m,j,\ell}(u) + o(\lambda^{-m})$$

$$= F_{r,m}(u,\lambda) + \sum_{p=r+1}^{m} \frac{f_{r+1,m,p}(u)}{\lambda^p} + o(\lambda^{-m}),$$

uniformly on each compact subset of (A,B) where

$$f_{r+1,m,p}(u) = \sum_{j+\ell=p} f_{r,m,j,\ell}(u).$$

Writing

$$F_{r+1,m}(u,\lambda) = \sum_{p=r+1}^{m} \frac{f_{r+1,m,p}(u)}{\lambda^p},$$

it follows that Theorem 1.4.7 holds for $r+1$.

This completes the proof.

1.5 SIMULTANEOUS APPROXIMATION

For the sake of completeness, in this section we state several results on the simultaneous approximation property of exponential type operators. Later in Chapter 4 we shall prove much more general results than those of this section and for this reason formal proofs of the results of this section are omitted.

THEOREM 1.5.1 : If $m \in \mathbb{N}$, $f \in D_\psi(A,B)$ and $f^{(m)}$ exists at a point $t \in (A,B)$, then

$$(1.5.1) \qquad \lim_{\lambda\to\infty} S_\lambda^{(m)}(f,t) = f^{(m)}(t).$$

Moreover, if $f^{(m)}$ exists and is continous on $\langle a,b\rangle$ then (1.5.1) holds uniformly in $t \in [a,b]$.

THEOREM 1.5.2 : Let $m \in \mathbb{N}$, $k \in \mathbb{N}^{o}$ and $f \in D_{\psi}(A,B)$. Then, if $f^{(2k+m+2)}$ exists at a point $t \in (A,B)$, there holds

$$(1.5.2)\quad \lim_{\lambda\to\infty} \lambda^{k+1}[S_{\lambda}^{(m)}(f,k,t)-f^{(m)}(t)] = \sum_{j=m}^{2k+m+2} Q(j,k,m,t)f^{(j)}(t)$$

and

$$(1.5.3)\quad \lim_{\lambda\to\infty} \lambda^{k+1}[S_{\lambda}^{(m)}(f,k+1,t)-f^{(m)}(t)] = o,$$

where $Q(j,k,m,t)$ are certain polynomials in t. Also, if $f^{(2k+m+2)}$ exists and is continuous on $\langle a,b\rangle$, then (1.5.2-3) hold uniformly in $t \in [a,b]$.

THEOREM 1.5.3 : Let $m \in \mathbb{N}$, $k \in \mathbb{N}^{o}$, $o \le p \le 2k+2$ and $f \in D_{\psi}(A,B)$. If $f^{(m+p)}$ exists and is continous on $\langle a,b\rangle$, then for all λ sufficiently large

$$(1.5.4)\quad \|S_{\lambda}^{(m)}(f,k,t)-f^{(m)}(t)\|_{C[a,b]} \le \max\{C\lambda^{-p/2}\,\omega(f^{(m+p)};\lambda^{-1}),\ C'\lambda^{-(k+1)}\},$$

where $C = C(k,m,p)$ and $C' = C'(k,m,p,f)$ and $\omega(f^{(m+p)};\delta)$ denotes the modulus of continuity of $f^{(m+p)}$ on $\langle a,b\rangle$.

The inverse and saturation theorems are also true in simultaneous approximation by $S_{\lambda}(f,k,t)$ and are as follows :

THEOREM 1.5.4 : Let $m \in \mathbb{N}$, $k \in \mathbb{N}^{o}$, $o < \alpha < 2$ and $f \in D_{\psi}(A,B)$. Then, in the following statements, the implications (i) $\Rightarrow$(ii) $\Leftrightarrow$(i
$\Rightarrow$ (iv) are valid :

(i) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\sup_{t \in [a_1,b_1]} |S^{(m)}_{\lambda_r}(f,k,t) - f^{(m)}(t)| = O(\lambda_n^{-\frac{\alpha(k+1)}{2}}) ;$$

(ii) $f^{(m)} \in \mathrm{Liz}\,(\alpha, k+1 \,;\, a_2, b_2)$;

(iii) (a) If $m' < \alpha(k+1) < m'+1$, $m' = o,1,2,\ldots,2k+1$, then $f^{(m'+m)}$ exists and $\in \mathrm{Lip}\,(\alpha(k+1)-m' ; a_2, b_2)$,
(b) If $\alpha(k+1) = m'+1$, $m' = o, 1,\ldots,2k$, then $f^{(m'+m)}$ exists and $\in \mathrm{Lip}^*(1; a_2, b_2)$;

(iv) $$\|S^{(m)}_{\lambda}(f,k,t) - f^{(m)}(t)\|_{C[a_3,b_3]} = O(\lambda^{-\frac{\alpha(k+1)}{2}}).$$

THEOREM 1.5.5 : If S_λ are regular, $m \in \mathbb{N}$, $k \in \mathbb{N}^o$ and $f \in D_\psi(A,B$ then, in the following statements, the implications (i) => (ii) => (iii) and (iv) => (v) => (vi) are true :

(i) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1,b_1]} |S^{(m)}_{\lambda_n}(f,k,t) - f^{(m)}(t)| = O(1) ;$$

(ii) $f^{(2k+m+1)} \in$ A.C. $[a_2, b_2]$ and $f^{(2k+m+2)} \in L_\infty[a_2, b_2]$;

(iii) $$\lambda^{k+1} \|S^{(m)}_{\lambda}(f,k,t) - f^{(m)}(t)\|_{C[a_3,b_3]} = O(1) ;$$

(iv) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1,b_1]} |S^{(m)}_{\lambda_n}(f,k,t) - f^{(m)}(t)| = o(1) ;$$

(v) $f \in C^{2k+m+2}[a_2, b_2]$ and $\sum_{i=m}^{2k+m+2} Q(i,k,m,t) f^{(i)}(t) = o$, $t \in [a_2, b_2]$ where $Q(i,k,m,t)$ are the polynomials occuring in (1.5.2) ;

(vi) $$\lambda^{k+1} \|S^{(m)}_{\lambda}(f,k,t) - f^{(m)}(t)\|_{C[a_3,b_3]} = o(1).$$

Finally we establish a Lorentz-type lemma for the derivatives of the kernels $W(\lambda,t,u)$ of exponential type operators.

LEMMA 1.5.6: For each $k \in \mathbb{N}^o$, there exist polynomials $q_{ij}^{[k]}(t)$ in t which do not depend on u or λ such that

$$\frac{\partial^k}{\partial t^k} W(\lambda,t,u) = Q_k(t,u,\lambda)\, W(\lambda,t,u),$$

where $Q_k(t,u,\lambda) = \sum_{\substack{2i+j\leq k\\ i,j\geq o}} \lambda^{i+j}(u-t)^j \dfrac{q_{ij}^{[k]}(t)}{(p(t))^k}$.

PROOF: Suppose that the result is true for k. Then by the induction hypothesis and (1.2.3)

$$\frac{\partial^{k+1}}{\partial t^{k+1}} W(\lambda,t,u)$$

$$= \frac{\partial}{\partial t}(Q_k(t,u,\lambda)\, W(\lambda,t,u))$$

$$= (\frac{\partial}{\partial t} Q_k(t,u,\lambda))\, W(\lambda,t,u) + Q_k(t,u,\lambda)\, (\frac{\partial}{\partial t} W(\lambda,t,u))$$

$$= (\frac{\partial}{\partial t} Q_k(t,u,\lambda))\, W(\lambda,t,u) + Q_k(t,u,\lambda)\, \frac{\lambda}{p(t)}(u-t)W(\lambda,t,u)$$

$$= W(\lambda,t,u)\left[(\frac{\partial}{\partial t} Q_k(t,u,\lambda)) + \frac{\lambda}{p(t)}(u-t)\, Q_k(t,u,\lambda) \right].$$

The last expression has the required form

$$Q_{k+1}(t,u,\lambda)\, W(\lambda,t,u) = \sum_{\substack{2i+j\leq k+1\\ i,j\geq o}} \lambda^{i+j}(u-t)^j \frac{q_{ij}^{[k+1]}(t)}{(p(t))^{k+1}}\, W(\lambda,t,u),$$

where

$$q_{ij}^{[k+1]}(t) = q_{i,j-1}^{[k]}(t) - k\, q_{ij}^{[k]}(t)p'(t) + \{(q_{ij}^{[k]}(t))' - (j+1)\, q_{i-1,j+1}^{[k]}(t)\}\, p(t),$$

$$2i+j \leq k+1,\quad i,j \geq o,$$

with the convention that $q_{ij}^{[k]}(t) \equiv o$ if any one of the constraints $2i + j \leq k$ and $i, j \geq o$ is violated.

Hence the result is true for k+1. For $k = o$, the result is trivial. Therefore, by induction, the result holds for all k.

CHAPTER 2

ORDINARY APPROXIMATION WITH GENERALIZED MICCHELLI COMBINATIONS

2.1 INTRODUCTION

Micchelli [28] considered the operators $T_{n,k}$ defined by $[I-(I-B_n)^k]$, where B_n are the Bernstein polynomials and obtained the following direct and saturation results :

(i) $||T_{n,k}(f)-f||_{C[0,1]} \leq \frac{3}{2}(2^k-1)\ \omega(f;n^{-1/2})$;

(ii) $||T_{n,k+1}(f)-f||_{C[0,1]} = O(n^{-(k+1)})$, if $f^{(2k+1)} \in \mathrm{Lip}(1;[0,1]$

(iii) If $|T_{n,k+1}(f,x)-f(x)| \leq \frac{Mx(1-x)}{2n^{k+1}} + o(n^{-(k+1)})$,

uniformly for $x \in [0,1]$ then $f,f',\ldots,f^{(2k+1)} \in C(0,1)$ and $A^k f$, where $Af(x) = \frac{1}{2}x(1-x)f''(x)$, has a continuous extension to $[0,1]$ whose derivative is in $\mathrm{Lip}_M 1$;

(iv) If $||T_{n,k+1}(f)-f||_{C[0,1]} = o(n^{-(k+1)})$, then f is linear on $[0,1]$.

The proofs of (iii)-(iv) were obtained by the semi-group method.

In this chapter, we study a generalized Micchelli type sequence of operators $S_{\lambda,k,m}(\cdot,t)$ constructed from certain linear combinations of iterates of S_λ and obtain somewhat more general results than the above. We also study inverse theorem for $S_{\lambda,k,m}(\cdot,t)$.

2.2 DEFINITIONS AND BASIC RESULTS

We define the linear combinations $S_{\lambda,k,m}$ of iterates of S_λ as follows :

$$(2.2.1)\qquad S_{\lambda,k,m}(f,t) = \sum_{r=1}^{k} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m}{k-r} S_\lambda^{r+m}(f,t)$$

for all $k \in \mathbb{N}$ and $m \in \mathbb{N}^o$, where $\beta(m,r) = \dfrac{\Gamma(m)\,\Gamma(r)}{\Gamma(m+r)}$ is the usual β function. It is easily seen that

$$(2.2.2)\qquad S_{\lambda,k,m}(f,t) = \sum_{r=m+1}^{k+m} (-1)^m \frac{(-1)^{r+1}\,\Gamma(r)}{\Gamma(m+1)\,\Gamma(r-m)} \binom{k+m}{r} S_\lambda^{r}(f,t).$$

Note that on taking $m = o$, $S_{\lambda,k,m}$ reduces to the Micchelli type combination $[\,I-(I-S_\lambda)^k]\,(f,t)$. Also we have

<u>LEMMA</u> 2.2.1 : Let $k \in \mathbb{N}$, $m \in \mathbb{N}^o$. Then

$$(2.2.3)\qquad \binom{k+m}{m} S_{\lambda,k,m+1} = \left[\binom{k+m+1}{m+1} S_{\lambda,k,m} - \binom{k+m}{m+1} S_{\lambda,k+1,m}\right].$$

<u>PROOF</u> : The identities

$$(2.2.4)\qquad \binom{n}{r}\binom{m}{s} - \binom{n}{s}\binom{m}{r} = \frac{n!\,m!}{r!\,s!\,(m+n-r-s)!}\left\{\binom{m+n-r-s}{n-r} - \binom{m+n-r-s}{m-r}\right\}$$

and

$$(2.2.5)\qquad \binom{n}{r} - \binom{n}{r-1} = \frac{n!}{r!\,(n-r+1)!}(n-2r+1)$$

can be easily verified. Using these and the definition of $S_{\lambda,k,m}$, we have

$$\binom{k+m+1}{m+1} S_{\lambda,k,m} - \binom{k+m}{m+1} S_{\lambda,k+1,m}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}(m+r-1)!}{m!\,(r-1)!} \left\{ \binom{k+m+1}{m+1}\binom{k+m}{k-r} - \binom{k+m}{m+1}\binom{k+m+1}{k-r+1} \right\} S_\lambda^{r+m}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}(m+r-1)!\,(k+m+1)!\,(k+m)!}{m!\,(r-1)!\,(m+1)!\,(m+r)!\,(2k-r)!} \left\{ \binom{2k-r}{k} - \binom{2k-r}{k-1} \right\} S_\lambda^{r+m}$$

$$= \sum_{r=2}^{k+1} \frac{(-1)^{r+2}(m+r-1)!\,(k+m-1)!\,(k+m)(r-1)}{m!\,(r-1)!\,(m+1)!\,(m+r)!\,k!\,(k-r+1)!} S_\lambda^{r+m}$$

$$= \binom{k+m}{m} \sum_{r=1}^{k} \frac{(-1)^{r+1}(m+r)!\,(k+m+1)!}{(m+1)!\,(r-1)!\,(k-r)!\,(m+r+1)!} S_\lambda^{r+m+1}$$

$$= \binom{k+m}{m} S_{\lambda,k,m+1},$$

which completes the proof.

For $k \in \mathbb{N}^o$, the k-th order moment $\mu_{\lambda,k}^{[p]}(t)$ of $S_\lambda^p (p \in \mathbb{N})$ is defined by

(2.2.6) $\quad \mu_{\lambda,k}^{[p]}(t) = S_\lambda^p\,((u-t)^k, t).$

It is clear that $\mu_{\lambda,k}^{[p]}(t)$ is a polynomial of degree $\leq k$ in t.

<u>LEMMA</u> 2.2.2: There holds the recursion relation

(2.2.7) $\quad \mu_{\lambda,k}^{[p+1]}(t) = \sum_{j=o}^{k} \binom{k}{j} \sum_{i=o}^{k-j} \frac{1}{i!} D^i [\mu_{\lambda,k-j}^{[p]}(t)]\ \mu_{\lambda,i+j}(t),$

where D denotes the operator $\frac{\partial}{\partial t}$, $p \in \mathbb{N}$ and $k \in \mathbb{N}^o$.

<u>PROOF</u> : By definition

$$\mu_{\lambda,k}^{[p+1]}(t) = S_\lambda(S_\lambda^p((u-t)^k, u_p), t)$$

$$= S_\lambda(S_\lambda^p((u-u_p+u_p-t)^k, u_p), t)$$

$$= \sum_{j=o}^{k} \binom{k}{j} S_\lambda((u_p-t)^j \, S_\lambda^p((u-u_p)^{k-j}, u_p), t)$$

$$= \sum_{j=o}^{k} \binom{k}{j} S_\lambda \Big(\sum_{i=o}^{k-j} \frac{(u_p-t)^{i+j}}{i!} D^i [\, \mu_{\lambda, k-j}^{[p]}(t) \,] \Big)$$

$$= \sum_{j=o}^{k} \binom{k}{j} \sum_{i=o}^{k-j} \frac{1}{i!} D^i [\, \mu_{\lambda, k-j}^{[p]}(t)] \quad \mu_{\lambda, i+j}(t),$$

completing the proof.

<u>COROLLARY</u> 2.2.3 : For $p \in \mathbb{N}$ and $k \in \mathbb{N}^o$ there holds

$$(2.2.8) \qquad \mu_{\lambda,k}^{[p]}(t) = O(\lambda^{-[k+1/2]}).$$

<u>PROOF</u> : For $p = 1$ the result is true from (1.3.3). Thus, assuming the result for a certain p we have to prove it for $p+1$. Then, since $\mu_{\lambda,k-j}^{[p]}(t) = O(\lambda^{-[\frac{k-j+1}{2}]})$ and $\mu_{\lambda,k-j}^{[p]}(t)$ is a polynomial in t of degree $\leq k-j$, it follows that also

$$D^i \, [\, \mu_{\lambda,k-j}^{[p]}(t) \,] = O(\lambda^{-[\frac{k-j+1}{2}]}).$$

Hence, by Lemma 2.2.2,

$$\mu_{\lambda,k}^{[p+1]}(t) = O\Big(\sum_{j=o}^{k} \sum_{i=o}^{k-j} \lambda^{-[\frac{k-j+1}{2}] - [\frac{i+j+1}{2}]} \Big)$$

$$= O\Big(\sum_{j=o}^{k} \sum_{i=o}^{k-j} \lambda^{-[\frac{k+i+1}{2}]} \Big)$$

$$= O(\lambda^{-[\frac{k+1}{2}]}),$$

which completes the proof.

<u>LEMMA</u> 2.2.4 : For $k, \ell \in \mathbb{N}$ and $m \in \mathbb{N}^o$,

$$(2.2.9) \qquad S_{\lambda,k,m}((u-t)^\ell, t) = O(\lambda^{-k}).$$

PROOF : Suppose the result holds for some m and for all k. Then by relation (2.2.3), it is clear that it also holds for m+1 and for all k. Therefore it is enough to show (2.2.9) for m = o and for all k.

For m = o, (2.2.9) reduces to

$$(2.2.10)\qquad S_{\lambda,k,o}\,((u-t)^{\ell},t) = [I-(I-S_{\lambda})^{k}]\,((u-t)^{\ell},t) = O(\frac{1}{\lambda^{k}}).$$

Suppose that (2.2.10) holds for some k. Then, by Lemma 2.2.2,

$$[\,I-(I-S_{\lambda})^{k+1}]\,((u-t)^{\ell},t) = \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}\,S_{\lambda}^{r}\,((u-t)^{\ell},t)$$

$$= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}\sum_{j=o}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}(\mu_{\lambda,\ell-j}^{[r-1]}(t))\,\mu_{\lambda,i+j}(t)$$

$$= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\sum_{j=o}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}(\mu_{\lambda,\ell-j}^{[r-1]}(t))\,\mu_{\lambda,i+j}(t)$$

$$+\sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}\sum_{j=o}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}(\mu_{\lambda,\ell-j}^{[r-1]}(t))\,\mu_{\lambda,i+j}(t)$$

$$=[\,I-(I-S_{\lambda})^{k}]\,((u-t)^{\ell},t)$$

$$+\sum_{j=o}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}(\sum_{r=1}^{k}(-1)^{r+2}\binom{k}{r}\mu_{\lambda,\ell-j}^{[r]}(t))\mu_{\lambda,i+j}(t)$$

$$= [\,I-(I-S_{\lambda})^{k}]\,((u-t)^{\ell},t)$$

$$-\sum_{j=o}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}([\,I-(I-S_{\lambda})^{k}]\,((u-t)^{\ell-j},t))\mu_{\lambda,i+j}(t)$$

$$= -\sum_{i=1}^{\ell}\frac{1}{i!}\,D^{i}(\,[\,I-(I-S_{\lambda})^{k}]\,((u-t)^{\ell},t))\,\mu_{\lambda,i}(t)$$

$$-\sum_{j=1}^{\ell}\binom{\ell}{j}\sum_{i=o}^{\ell-j}\frac{1}{i!}\,D^{i}(\,[\,I-(I-S_{\lambda})^{k}]\,((u-t),t))\,\mu_{\lambda,i+j}(t)$$

$$= O(\lambda^{-(k+1)})\ \text{(by Corollary 2.2.3)}.$$

Thus the result (2.2.10) also holds for k+1. For $k = 1$, it is true in view of Lemma 1.3.3. Hence, by induction it holds for all k.

This completes the proof of Lemma 2.2.4.

Throughout this chapter Ψ denotes a common GTF for the operators $S_\lambda^{m+1}, S_\lambda^{m+2}, \ldots, S_\lambda^{k+m+1}$. We note that the function $(1+t^2)^N$, $N > o$ is always such a GTF.

In the following theorem we prove that $S_{\lambda,k,m}(\cdot,t)$ is an approximation process for all functions belonging to $D_\psi(A,B)$.

<u>THEOREM</u> 2.2.5 : If $f \in D_\psi(A,B)$ and is continuous at a point $t \in (A,B)$, then

$$(2.2.11) \qquad \lim_{\lambda \to \infty} S_{\lambda,k,m}(f,t) = f(t).$$

Further, if f is continuous on $\langle a,b \rangle$ then (2.2.11) holds uniformly on $[a,b]$.

<u>PROOF</u> : To prove the theorem, it is sufficient to show that

(a) $S_{\lambda,k,m}(1,t) = 1$ and

(b) $\lim_{\lambda \to \infty} S_\lambda^r(f,t) = f(t)$, $r = m+1, m+2, \ldots, k+m$,

if t is a continuity point of f and uniformly on $[a,b]$ if f is continuous on $\langle a,b \rangle$.

To prove (a), assuming it to be true for a certain m and all k, by Lemma 2.2.1 we have

$$S_{\lambda,k,m+1}(1,t) = \binom{k+m}{m}^{-1} \left[\binom{k+m+1}{m+1} - \binom{k+m}{m+1} \right] = 1,$$

and therefore it holds for m+1 and all k. Hence it remains to show that it holds for m = o and all k. But

$$S_{\lambda,k,o}(1,t) = [I-(I-S_\lambda)^k]\ (1,t) = 1,$$

since $S_\lambda^p(1,t) = 1$ for all $p \in IN$.

The proof of the statement (b) easily follows from Corollary 2.2.3 and the Schwarz inequality along lines similar to those of the proof of (1.4.7).

2.3 DIRECT THEOREMS

First of all we establish an asymptotic formula for the operator $S_{\lambda,k,m}$.

THEOREM 2.3.1 : Let $f \in D_\psi(A,B)$. If $f^{(2k)}$ exists at a point $t \in (A,B)$, then

$$(2.3.1)\qquad \lim_{\lambda\to\infty} \lambda^k [S_{\lambda,k,m}(f,t)-f(t)] = \sum_{j=2}^{2k} Q(j,k,m,t)f^{(j)}(t)$$

and

$$(2.3.2)\qquad \lim_{\lambda\to\infty} \lambda^k [S_{\lambda,k+1,m}(f,t)-f(t)] = o,$$

where $Q(j,k,m,t)$ are certain polynomials in t.

Further, if $f^{(2k)}$ exists and is continuous on $\langle a,b\rangle$ then (2.3.1-2) hold uniformly on $[a,b]$.

PROOF : By the Taylor expansion of f(u) about u = t, we have

$$\lambda^k [S_{\lambda,k,m}(f,t)-f(t)]$$

$$= \lambda^k \sum_{\ell=1}^{2k} \frac{f^{(\ell)}(t)}{\ell!} S_{\lambda,k,m}((u-t)^\ell,t)$$

$$+ \lambda^k \sum_{r=1}^{k} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m}{k-r} S_\lambda^{r+m}(\varepsilon(u,t)(u-t)^{2k},t)$$

$= I_1 + I_2$, say, where $\varepsilon(u,t) \to 0$ as $u \to t$.

Let us estimate I_1 first. Since $S_{\lambda,k,m}(u,t) = t$ (which follows along the lines of the proof of $S_{\lambda,k,m}(1,t) = 1$), by Lemma 2.2.4

$$I_1 = \lambda^k \sum_{\ell=2}^{2k} \frac{f^{(\ell)}(t)}{\ell!} S_{\lambda,k,m}((u-t)^\ell, t)$$

$$= \sum_{\ell=2}^{2k} Q(\ell,k,m,t) f^{(\ell)}(t) + o(1), \text{ where } Q(\ell,k,m,t)$$

is the coefficient of λ^{-k} in $S_{\lambda,k,m}((u-t)^\ell, t)$.

Since $\epsilon(u,t) \to 0$ as $u \to t$, so for a given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|\epsilon(u,t)| < \epsilon$ whenever $|u-t| \leq \delta$.

Now, if $\chi_\delta(u)$ is the characteristic function of the interval $(t-\delta, t+\delta)$, we have

$$|I_2| \leq \lambda^k \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} S_\lambda^{r+m}(|\epsilon(u,t)|(u-t)^{2k} \chi_\delta(u), t)$$

$$+ \lambda^k \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} S_\lambda^{r+m}(|\epsilon(u,t)|(u-t)^{2k}(1-\chi_\delta(u)), t)$$

$$= I_3 + I_4, \text{ say.}$$

Then,

$$I_3 \leq \sup_{|u-t| \leq \delta} |\epsilon(u,t)| \lambda^k \Big[\sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} \Big] \max_{1 \leq r \leq k} [S_\lambda^{r+m}((u-t)^{2k}, t|$$

$$< \epsilon \lambda^k \frac{M}{\lambda^k} = M\epsilon \text{ (by Corollary 2.2.3).}$$

Also, for an arbitrary $p(\in \mathbb{N}) > k$, using Schwarz inequality and Corollary 2.2.3,

$$I_4 \leq \lambda^k \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} S_\lambda^{r+m}\left(\frac{M'\Psi(u)(u-t)^{2p}}{\delta^{2p-2k}}, t\right)$$

$$\leq \frac{M''}{\lambda^{p-k}}.$$

Thus $I_4 = o(1)$ and therefore in view of the arbitrariness of $\epsilon > o$ we have $|I_2| = o(1)$.

Combining these estimates of I_1 and I_2, (2.3.1) follows. The assertion (2.3.2) can be proved along similar lines by noting that $S_{\lambda,k+1,m}((u-t)^{\ell},t) = O(\lambda^{-(k+1)})$.

The second assertion follows due to the uniform continuity of $f^{(2k)}$ on $[a,b]$ (enabling δ to become independent of $t \in [a,b]$) and the uniformness of $o(1)$ term in the estimate of I_1(because, in fact it is a polynomial in λ^{-1} and t).

This completes the proof.

In the next result we obtain an estimate of the degree of approximation by $S_{\lambda,k,m}$ for smooth functions.

<u>THEOREM</u> 2.3.2 : Let $o \leq p \leq 2k$, $f \in D_{\psi}(A,B)$ and $f^{(p)}$ exist and be continuous on $\langle a,b \rangle$. Then, for all λ sufficiently large, there holds

$$(2.3.3) \qquad \|S_{\lambda,k,m}(f,t)-f(t)\|_{C\,[a,b]} \leq \max\{C\,\lambda^{-p/2}\,\omega(f^{(p)};\lambda^{-1/2}),\; C'\lambda^{-k}\},$$

where $C = C(k,m,p)$, $C' = C'(k,m,p,f)$ and $\omega(f^{(p)};\delta)$ denotes the modulus of continuity of $f^{(p)}$ on $\langle a,b \rangle$.

<u>PROOF</u> : Proceeding as in the proof of Theorem 1.3.9, by Corollary 2.2.3 and Schwarz inequality we have

$$(2.3.4) \qquad |S_{\lambda,k,m}(F(u,t),t)| \leq C_1\,\lambda^{-p/2}\,\omega(f^{(p)};\lambda^{-1/2}) + C_3\,\lambda^{-(k+1)},$$

where $F(u,t)$ is defined by (1.3.11) and C_1 does not depend on f.

In view of Lemma 2.2.4

$$(2.3.5) \qquad S_{\lambda,k,m}\Big(\sum_{j=1}^{p} \frac{f^{(j)}(t)}{j!}(u-t)^j, t\Big) = O(\lambda^{-k}),$$

uniformly in $t \in [a,b]$.

Now (2.3.3) follows from (2.3.4-5) completing the proof.

2.4 INVERSE THEOREM FOR $S_{\lambda,k+1,m}(\cdot,t)$

This section is devoted to the proof of the following inverse theorem for the operators $S_{\lambda,k+1,m}$ ($k \in \mathbb{N}^o$).

THEOREM 2.4.1 : If $o < \alpha < 2$ and $f \in D_\psi(A,B)$, then, in the following statements, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) hold :

(i) $\displaystyle\sup_{t \in [a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = O(\lambda_n^{-\alpha(k+1)/2})$;

(ii) $f \in Liz(\alpha,k+1;a_2,b_2)$;

(iii) (a) If $m' < \alpha(k+1) < m'+1$, $m' = o,1,\dots,2k+1$, then $f^{(m')}$ exists and $\in Lip\,(\alpha(k+1)-m'\ ;\ a_2,b_2)$,

(b) If $\alpha(k+1) = m'+1$, $m' = o,1,\dots, 2k$, then $f^{(m')}$ exists and $\in Lip^*\,(1;a_2,b_2)$;

(iv) $||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2})$.

Before starting the proof of Theorem 2.4.1, we first prove two auxiliary results.

LEMMA 2.4.2 : Let $A < a < a' < a'' < b'' < b' < b < B$. If $f \in C_o$ with supp $f \subset [a'',b'']$ and

$$(2.4.1) \qquad ||S_{\lambda_n,k+1,m}(f,t)-f(t)||_{C[a,b]} \le M\, \lambda_n^{-\alpha(k+1)/2},$$

$n = 1,2,\ldots$ where M is independent of n, then for all λ sufficiently large,

$$(2.4.2)\quad K(\xi,f;a',b') \leq M_o[\lambda^{-\alpha(k+1)/2} + \lambda^{k+1}\,\xi\,K(\lambda^{-(k+1)}, f;a',b')],$$

where M_o is a constant. Consequently, $K(\xi,f;a',b') \leq M_1\,\xi^{\alpha/2}$, for some constant M_1 i.e., $f \in C_o(\alpha,k+1;a',b')$.

PROOF : To establish (2.4.2) it is sufficient to show that

$$(2.4.3)\quad K(\xi,f;a',b') \leq M_o[\lambda_n^{-\alpha(k+1)/2} + \lambda_n^{k+1}\,\xi\,K(\lambda_n^{-(k+1)},f;a',b')],$$

for all n sufficiently large. This is because for each λ sufficiently large we can choose λ_{n-1} and λ_n such that $\lambda_{n-1} < \lambda \leq \lambda_n$

Now, since $[a",b"] \subset (a',b')$ therefore we can choose a $\delta > 0$ such that $(a"-2\delta,b"+2\delta) \subset [a',b']$. Let $g \in C^\infty$ be such that $g(t) = 1$ on $[a"-2\delta,b"+2\delta]$ and $g(t) = 0$ on $(A,B)\setminus(a',b')$.

Defining h_n to be equal to $g(t)\,S_{\lambda_n,k+1,m}(f,t)$, $h_n \in G(a',b')$

Let S denote the set $[a,b]\setminus(a"-2\delta,b"+2\delta)$. It $t \in S$ and $\chi_\delta(u)$ denotes the characteristic function of $(t-\delta,t+\delta)$ then, since supp $f \subset [a",b"]$, for all $u \in (A,B)$

$$f(u) = (1-\chi_\delta(u))f(u).$$

Hence, for any $p\ (\in \mathbb{N}) \geq 2k+2$ we have

$$\begin{aligned}|f(u)| &= (1-\chi_\delta(u))\,|f(u)| \\ &\leq (\tfrac{u-t}{\delta})^{2p}\,|f(u)| \\ &\leq M_1(\tfrac{u-t}{\delta})^{2p}\,\Upsilon(u),\end{aligned}$$

where M_1 is a constant.

Now, with $t \in S$ and $\ell = o,1,2,\ldots,2k+2$, by Lemma 1.5.6

$$|S^{(\ell)}_{\lambda,k+1,m}(f,t)|$$

$$\leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \int_A^B |W^{(\ell)}(\lambda,t,u_{r+m-1})|.$$

$$\cdot S_\lambda^{r+m-1}(|f(u)|,u_{r+m-1})du_{r+m-1}$$

$$\leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{\substack{2r'+s' \leq \ell \\ r',s' \geq o}} \lambda^{r'+s'} \frac{|q^{[\ell]}_{r's'}(t)|}{(p(t))^{\ell}} \int_A^B W(\lambda,t,u_{r+m-1})$$

$$\cdot |u_{r+m-1}-t|^{s'} S_\lambda^{r+m-1}\left(\frac{M}{\delta^{2p}}(u-t)^{2p}\psi(u),u_{r+m-1}\right)du_{r+m-1}.$$

But, by Schwarz inequality

$$\int_A^B W(\lambda,t,u_{r+m-1})\ |u_{r+m-1}-t|^{s'} S_\lambda^{r+m-1}((u-t)^{2p}\psi(u),u_{r+m-1})du_{r+m-1}$$

$$\leq \left(\int_A^B W(\lambda,t,u_{r+m-1})\ S_\lambda^{r+m-1}(\psi^2(u),u_{r+m-1})du_{r+m-1}\right)^{1/2}\cdot$$

$$\cdot\left(\int_A^B W(\lambda,t,u_{r+m-1})\ (u_{r+m-1}-t)^{2s'}\ S_\lambda^{r+m-1}((u-t)^{4p},u_{r+m-1})du_{r+m-1}\right)^{1/2}$$

$$= (S_\lambda^{r+m}(\psi^2(u),t))^{1/2}\cdot$$

$$\cdot\left(\int_A^B W(\lambda,t,u_{r+m-1})(u_{r+m-1}-t)^{2s'}\ S_\lambda^{r+m-1}((u-t)^{4p},u_{r+m-1})du_{r+m-1}\right)^{1/2}$$

Now, by Corollary 2.2.3,

$$\int_A^B W(\lambda,t,u_{r+m-1})(u_{r+m-1}-t)^{2s'}\ S_\lambda^{r+m-1}((u-t)^{4p},u_{r+m-1})du_{r+m-1}$$

$$= \sum_{\ell_1=o}^{4p} \binom{4p}{\ell_1} \sum_{\ell_2=o}^{4p-\ell_1} \frac{1}{\ell_2!} \int_A^B W(\lambda,t,u_{r+m-1})(u_{r+m-1}-t)^{2s'+\ell_1+\ell_2}.$$

$$\cdot D^{\ell_2}(\mu^{[r+m-1]}_{\lambda,4p-\ell_1}(t))du_{r+m-1}$$

$$= \sum_{\ell_1=0}^{4p} \binom{4p}{\ell_1} \sum_{\ell_2=0}^{4p-\ell_1} \frac{1}{\ell_2!} D^{\ell_2}(\mu^{[r+m-1]}_{\lambda,4p-\ell_1}(t))\,\mu_{\lambda,2s'+\ell_1+\ell_2}(t)$$

$$= O\Big(\sum_{\ell_1=0}^{4p} \sum_{\ell_2=0}^{4p-\ell_1} \lambda^{-[\frac{4p-\ell_1+1}{2}]-[\frac{2s'+\ell_1+\ell_2+1}{2}]}\Big)$$

$$= O\Big(\sum_{\ell_1=0}^{4p} \sum_{\ell_2=0}^{4p-\ell_1} \lambda^{-[\frac{4p+2s'+\ell_2+1}{2}]}\Big)$$

$$= O\big(\lambda^{-[\frac{4p+2s'+1}{2}]}\big),$$

uniformly in $t \in S$.

It follows that uniformly for all $t \in S$

$$|S^{(\ell)}_{\lambda_n,k+1,m}(f,t)| = O(\lambda_n^{-(k+1)}),\ \ell = 0,1,\ldots,2k+2.$$

Consequently, since $h_n(t)$ coincides with $S_{\lambda_n,k+1,m}(f,t)$ on $[a''-2\delta, b''+2\delta]$ we have for $i = 0$ and $2k+2$

$$(2.4.4)\quad \|h_n^{(i)}(t) - S^{(i)}_{\lambda_n,k+1,m}(f,t)\|_{C[a,b]} \le M_2\lambda_n^{-(k+1)},$$

for all n sufficiently large.

Since $h_n \in G(a',b')$ therefore by (2.4.4)

$$K(\xi,f;a',b') \le \{\|f-h_n\|_{C[a',b']} + \xi(\|h_n\|_{C[a',b']} + \|h_n^{(2k+2)}\|_{C[a',b']})\}$$

$$\le \{\|f(t)-S_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']} + \|S_{\lambda_n,k+1,m}(f,t)-h_n(t)\|_{C[a',b']}$$

$$+\xi(\|h_n(t)-S_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']} + \|S_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']}$$

$$+\|h_n^{(2k+2)}(t)-S^{(2k+2)}_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']} + \|S^{(2k+2)}_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']})$$

$$\le 3M_2\lambda_n^{-(k+1)} + \|f(t)-S_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']}$$

$$+\xi[\|S_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']} + \|S^{(2k+2)}_{\lambda_n,k+1,m}(f,t)\|_{C[a',b']}]$$

Now, for each $g \in G(a',b')$ we have

$$||S_{\lambda_n,k+1,m}(f,t)||_{C[a',b']} \leq ||S_{\lambda_n,k+1,m}(f-g,t)||_{C[a',b']} + ||S_{\lambda_n,k+1,m}(g,t)||_{C[a',b']}$$

$$\leq M_3||f-g||_{C[a',b']} + M_4||g||_{C[a',b']},$$

since supp $(f-g) \subset [a',b']$ and supp $g \subset [a',b']$, where M_3 and M_4 are certain constants.

Hence, it is enough to show that there exists an M_5 such that for each $g \in G(a',b')$,

$$||S^{(2k+2)}_{\lambda,k+1,m}(f,t)||_{C[a',b']} \leq M_5\lambda^{k+1}\{ ||f-g||_{C[a',b']} + \lambda^{-(k+1)}||g^{(2k+2)}||_{C[a',b']} \}.$$

We can write

$$||S^{(2k+2)}_{\lambda,k+1,m}(f,t)||_{C[a',b']}$$

$$\leq ||\sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \frac{d^{2k+2}}{dt^{2k+2}} S_\lambda^{r+m}(f-g,t)||_{C[a',b']}$$

$$(2.4.5) \qquad + ||\sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \frac{d^{2k+2}}{dt^{2k+2}} S_\lambda^{r+m}(g,t)||_{C[a',b']}$$

$$= I_1 + I_2, \text{ say.}$$

We shall first estimate I_1. By Lemma 1.5.6

$$I_1 \leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \cdot$$

$$\cdot|| \frac{d^{2k+2}}{dt^{2k+2}} S_\lambda(S_\lambda^{r+m-1}(f(u)-g(u),u_{r+m-1}),t)||_{C[a',b']}$$

$$\leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}\Big\| \sum_{\substack{2i+j\leq 2k+2\\ i,j\geq o}} \lambda^{i+j} \frac{|q_{ij}^{[2k+2]}(t)|}{(p(t))^{2k+2}} \cdot$$

$$\cdot \int_A^B W(\lambda, t, u_{r+m-1})|u_{r+m-1}-t|^j\, S_\lambda^{r+m-1}(|(f-g)(u)|, u_{r+m-1})du_{r+m-1}\Big\|_{C[a',b']}$$

$$\leq \|f-g\|_{C[a',b']} \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1} \sum_{\substack{2i+j\leq 2k+2\\ i,j\geq o}} \lambda^{i+j} \cdot$$

$$\cdot\Big\|\frac{|q_{ij}^{[2k+2]}(t)|}{(p(t))^{2k+2}} \int_A^B W(\lambda,t,u_{r+m-1})|u_{r+m-1}-t|^j\, du_{r+m-1}\Big\|_{C[a',b']},$$

since supp $f \cup$ supp $g \subset [a',b']$.

Hence, by Schwarz inequality and Corollary 2.2.3

$$I_1 \leq M_6 \lambda^{k+1}\|f-g\|_{C[a',b']},$$

where M_6 is a constant independent of g.

To estimate I_2, first notice that

(2.4.6) $\quad \int_A^B W^{(k)}(\lambda,t,u)(u-t)^i du = o$ for $k > i$.

$$I_2 \leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}\Big\|\int_A^B W^{(2k+2)}(\lambda,t,u_{r+m-1})\cdot$$

$$\cdot\, S_\lambda^{r+m-1}(g(u), u_{r+m-1})du_{r+m-1}\Big\|_{C[a',b']}.$$

Now, by Taylor's expansion of g, we have

$$\Big\|\int_A^B W^{(2k+2)}(\lambda,t,u_{r+m-1}) S_\lambda^{r+m-1}(g(u),u_{r+m-1})du_{r+m-1}\Big\|_{C[a',b']}$$

$$= \Big\|\int_A^B W^{(2k+2)}(\lambda,t,u_{r+m-1}) S_\lambda^{r+m-1}\Big(\sum_{j=o}^{2k+1} \frac{g^{(j)}(t)}{j!}(u-t)^j$$

$$+ \frac{g^{(2k+2)}(\xi)}{(2k+2)!}(u-t)^{2k+2}, u_{r+m-1}\Big)\, du_{r+m-1}\Big\|_{C[a',b']}$$

(ξ lies between u and t)

$$= \left\| \int_A^B W^{(2k+2)}(\lambda, t, u_{r+m-1}) \cdot \right.$$

$$\left. \cdot\, S_\lambda^{r+m-1}\left(\frac{g^{(2k+2)}(\xi)}{(2k+2)!}(u-t)^{2k+2}, u_{r+m-1}\right) du_{r+m-1} \right\|_{C[a',b']} \quad \text{(by (2.4.6)}$$

$$\leq \sum_{\substack{2i+j \leq 2k+2 \\ i,j \geq 0}} \lambda^{i+j} \left\| \frac{|q_{ij}^{[2k+2]}(t)|}{(p(t))^{2k+2}} \int_A^B W(\lambda, t, u_{r+m-1}) |u_{r+m-1}-t|^j \cdot \right.$$

$$\left. \cdot\, S_\lambda^{r+m-1}\left(\frac{|g^{(2k+2)}(\xi)|}{(2k+2)!}(u-t)^{2k+2}, u_{r+m-1}\right) du_{r+m-1} \right\|_{C[a',b']}$$

(by Lemma 1.5.6)

$$\leq \frac{1}{(2k+2)!} \|g^{(2k+2)}\|_{C[a',b']} \sum_{\substack{2i+j \leq 2k+2 \\ i,j \geq 0}} \lambda^{i+j} \left\| \frac{|q_{ij}^{[2k+2]}(t)|}{(p(t))^{2k+2}} \cdot \right.$$

$$\cdot \left(\int_A^B W(\lambda, t, u_{r+m-1})(u_{r+m-1}-t)^{2j} du_{r+m-1} \right)^{1/2} \cdot \left(\int_A^B W(\lambda, t, u_{r+m-1}) \right.$$

$$\left. \left. \cdot\, S_\lambda^{r+m-1}((u-t)^{2(2k+2)}, u_{r+m-1}) du_{r+m-1} \right)^{1/2} \right\|_{C[a',b']}$$

(applying Schwarz inequality twice)

$$\leq M_7 \|g^{(2k+2)}\|_{C[a',b']}, \text{ by Corollary 2.2.3.}$$

Hence,

$$I_2 \leq M_8 \|g^{(2k+2)}\|_{C[a',b']} .$$

Substituting the estimates of I_1 and I_2 into (2.4.5) we obtain (2.4.3).

The rest of the proof follows along Lorentz [5].

<u>LEMMA</u> 2.4.3 : Let $a < a' < a'' < b'' < b' < b$. If $f \in C_o$ with supp $f \subset [a'',b'']$ and $f \in C_o(\alpha,k+1;a',b')$ then, for all λ sufficiently large

$$(2.4.7) \qquad ||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a,b]} \le M\lambda^{-\alpha(k+1)/2}.$$

<u>PROOF</u> : For $g \in G(a',b')$, we have

$$||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a,b]}$$

$$\le ||S_{\lambda,k+1,m}(f-g,t)||_{C[a,b]} + ||S_{\lambda,k+1,m}(g,t)-f(t)||_{C[a,b]}$$

$$= I_1+I_2, \text{ say.}$$

Clearly, $I_1 \le M_1||f-g||_{C[a',b']}$, since supp$(f-g)\subset[a',b']$.

We estimate I_2 as follows : By Theorem 2.3.1, we have

$$I_2 \le ||g-f||_{C[a',b']} + ||S_{\lambda,k+1,m}(g,t)-g(t)||_{C[a',b']}$$

$$\le ||g-f||_{C[a',b']} + M_2\lambda^{-(k+1)} \sum_{j=2}^{2k+2} ||g^{(j)}||_{C[a',b']}$$

$$\le ||g-f||_{C[a',b']} + M_3\lambda^{-(k+1)}(||g||_{C[a',b']} + ||g^{(2k+2)}||_{C[a',b']}$$

where M_3 is another constant.

Thus, since $f \in C_o(\alpha,k+1;a',b')$ we have

$$||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a,b]} \le M_4K(\lambda^{-(k+1)},f;a',b') \le M_5\lambda^{-\alpha(k+1)/2}.$$

This completes the proof.

PROOF OF THEOREM 2.4.1 : To prove the theorem it is enough to show that the implications (ii) $\Rightarrow$(iv) and (i)$\Rightarrow$ (ii) hold, the equivalence of (ii) and (iii) being well known. First let us assume (ii). Let a', a'', b', b'' be such that $a_2 < a' < a'' < a_3$ and $b_3 < b'' < b' < b_2$. Also, let $g \in C_0^\infty$ be such that $g(t) = 1$ on $[a'', b'']$ and supp $g \subset [a', b']$. Then, since $f \in \mathrm{Liz}(\alpha, k+1; a_2, b_2)$, also $fg \in \mathrm{Liz}(\alpha, k+1; a_2, b_2)$. Now, since supp $fg \subset (a_2, b_2)$ by Theorem 1.3.6 and Lemma 2.4.3

$$\|S_{\lambda,k+1,m}(fg,t) - fg(t)\|_{C[a_2,b_2]} = O(\lambda^{-\alpha(k+1)/2}).$$

Consequently, since $g(t) = 1$ on $[a'', b'']$ and $[a_3, b_3] \subset (a'', b'')$

$$(2.4.8)\quad \|S_{\lambda,k+1,m}(fg,t) - f(t)\|_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}).$$

Hence, it is enough to show that

$$(2.4.9)\quad \|S_{\lambda,k+1,m}(fg,t) - S_{\lambda,k+1,m}(f,t)\|_{C[a_3,b_3]} = o(\lambda^{-(k+1)}).$$

For this, if we choose $\delta \le \min\{a_3 - a'', b'' - b_3\}$ and let $\chi(u)$ denote the characteristic function of $(A,B) \setminus (a'', b'')$, then

$$\|S_{\lambda,k+1,m}((fg-f)(u),t)\|_{C[a_3,b_3]}$$

$$= \|S_{\lambda,k+1,m}((fg-f)(u)\,\chi(u),t\|_{C[a_3,b_3]}$$

$$\le \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \|S_\lambda^{r+m}(|(fg-f)(u)|\,\chi(u),t\|_{C[a_3,b_3]}$$

$$\le \Big[\sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1}\Big] \max_{1\le r\le k+1} \|S_\lambda^{r+m}\Big(\frac{M\psi(u)\chi(u-t)^{2p}}{\delta^{2p}}, t\Big)\|_{C[a_3,b_3]}$$

$\leq \frac{M}{\lambda^p}$ (by Schwarz inequality and Corollary 2.2.3), where $p(\in \mathbb{N}) > k+1$ is arbitrary.

Thus, (2.4.9) follows.

Consequently,

$$||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}),$$

proving the implication (ii) $\Rightarrow$ (iv).

Next, let us assume (i). Then since $S_{\lambda_n}f$ are C^∞ functions it is clear that $f \in C[a_1,b_1]$ and therefore $\sup_{t\in[a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = ||S_{\lambda_n,k+1,m}(f,t)-f(t)||_{C[a_1,b_1]}$. We put $\tau = \alpha$

First we consider the case when $o < \tau \leq 1$. Let $a_1 < a' < a'' < a_2$ and $b_2 < b'' < b' < b_1$. Also, let $g \in C_0^\infty$ be such that supp $g \subset [a'',b'']$ and $g(t) = 1$ on $[a_2,b_2]$. Then, with $\chi_1(u)$ denoting the characteristic function of $[a_1,b_1]$, for $t \in [a',b']$, we have

$$\begin{aligned} &S_{\lambda_n,k+1,m}(fg,t)-fg(t) \\ &= S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)+S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t)),t) \\ (2.4.10)\quad &= S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)+S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t))\chi_1(u), \\ &\quad + o(\lambda_n^{-(k+1)}) \text{ (by Theorem 2.3.1)} \\ &= I_1+I_2+o(\lambda_n^{-(k+1)}), \text{ say,} \end{aligned}$$

where the o-term holds uniformly for $t \in [a',b']$.

In view of (i), we have

$$(2.4.11)\qquad ||I_1||_{C[a',b']} \leq ||g||_{C[a',b']}\ ||S_{\lambda_n,k+1,m}(f,t)-f(t)||_{C[a',}$$

$$= O(\lambda_n^{-\tau/2}).$$

Next, by the mean value theorem

$$I_2 = S_{\lambda_n,k+1,m}(f(u)g'(\xi)(u-t)\ \chi_1(u),t),$$

where ξ lies between u and t. Then by Schwarz inequality and Corollary 2.2.3

$$||I_2||_{C[a',b']}$$

$$\leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}||S_{\lambda_n}^{r+m}(|f(u)||g'(\xi)||u-t|\ \chi_1(u),t)||_{C[a',b']}$$

$$\leq ||g'||_{C[a',b']}\ ||f||_{C[a_1,b_1]}\left[\sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}\right].$$

$$\cdot \max_{1\leq r\leq k+1} ||S_{\lambda_n}^{r+m}((u-t)^2,t)||_{C[a',b']}^{1/2}$$

$$\leq M\lambda_n^{-1/2}.$$

In particular,

$$(2.4.12)\qquad ||I_2||_{C[a',b']} = O(\lambda_n^{-\tau/2}).$$

Combining (2.4.10), (2.4.11) and (2.4.12) we conclude that

$$||S_{\lambda_n,k+1,m}(fg,t)-fg(t)||_{C[a',b']} = O(\lambda_n^{-\tau/2}).$$

From this in view of Lemma 2.4.2, $fg \in C_o(\alpha,k+1;a',b')$ and hence due to Theorem 1.3.6, $fg \in Liz(\alpha,k+1;a',b')$ which implies that $f \in Liz(\alpha,k+1;a_2,b_2)$ since $g(t) = 1$ on $[a_2,b_2]$. Thus (ii) follows.

Hence the implication (i)$\Longrightarrow$(ii) holds if $o<\tau\leq 1$. Thus, to prove the implication for $o<\tau<2k+2$, it is sufficient to assume it for $\tau \in (p-1,p)$ and prove it for $\tau \in [p,p+1)$ $(p = 1,2,\ldots,2k+1)$. Hence we assume that $\tau \in [p,p+1)$ and (i) holds. Then, in view of the assumption (i)$\Rightarrow$(ii) for the interval $(p-1,p)$ and the equivalence of (ii) and (iii), it follows that $f^{(p-1)}$ exists and $\in Lip(1-\delta;a_1^*,b_1^*)$, for any interval $[a_1^*,b_1^*]\subset(a_1,b_1)$ and $\delta>o$. Let a_2^*,b_2^* be such that $[a_2,b_2]\subset(a_2^*,b_2^*$ and $[a_2^*,b_2^*]\subset(a_1^*,b_1^*)$. Let $g \in C_o^\infty$ be such that $g(t) = 1$ on $[a_2,b_2]$ and supp $g\subset(a_2^*,b_2^*)$. Then, with $\chi_2(u)$ denoting the characteristi function of the interval $[a_1^*,b_1^*]$, we have

$$|| S_{\lambda_n,k+1,m}(fg,t)-f(t)g(t)||_{C[a_2^*,b_2^*]}$$

(2.4.13) $$\leq ||S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)||_{C[a_2^*,b_2^*]}$$

$$+ ||S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t))\chi_2(u),t)||_{C[a_2^*,b_2^*]} + o(\lambda_n^{-(k}$$

by an application of Theorem 2.3.1 to the function $f(u)(g(u)-g(t))(1-\chi_2(u))$. Now

$$(2.4.14)\quad \begin{aligned} &\|S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)\|_{C[a_2^*,b_2^*]} \\ &\le \|g\|_{C[a_2^*,b_2^*]} \|S_{\lambda_n,k+1,m}(f,t)-f(t)\|_{C[a_1,b_1]} \\ &= O(\lambda_n^{-\tau/2}). \end{aligned}$$

Also, using Taylor's expansion of f

$$\begin{aligned} &\|S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t))\,\chi_2(u),t)\|_{C[a_2^*,b_2^*]} \\ &= \|S_{\lambda_n,k+1,m}\Big(\Big[\sum_{i=0}^{p-1}\frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(p-1)}(\xi)-f^{(p-1)}(t)}{(p-1)!}(u- \\ &\cdot(g(u)-g(t))\,\chi_2(u),t\Big)\|_{C[a_2^*,b_2^*]} \quad (\xi \text{ lying between } u \text{ and } t \end{aligned}$$

$$(2.4.15)\quad \begin{aligned} &\le \|S_{\lambda_n,k+1,m}\Big(\sum_{i=0}^{p-1}\frac{f^{(i)}(t)}{i!}(u-t)^i(g(u)-g(t))\,\chi_2(u),t\Big)\|_{C[a_2^*,} \\ &+ \frac{M}{(p-1)!}\|g\|_{C[a_2^*,b_2^*]}\sum_{r=1}^{k+1}\frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}\cdot \\ &\cdot \|S_{\lambda_n}^{r+m}(|u-t|^{p+1-\delta},t)\|_{C[a_2^*,b_2^*]}, \\ &= I_1 + I_2, \text{ say,} \end{aligned}$$

where M is the $\mathrm{Lip}(1-\delta;a_1^*,b_1^*)$ constant for $f^{(p-1)}$. In view of Lemma 2.2.4

$$(2.4.16)\quad I_1 = O(\lambda_n^{-(k+1)}).$$

Also, by Corollary 2.2.3 and Hölder's inequality

$$(2.4.17)\quad \begin{aligned} I_2 &= O(\lambda_n^{-(p+1-\delta)/2}) \\ &= O(\lambda_n^{-\tau/2}), \end{aligned}$$

on having chosen $o < \delta \leq p+1-\gamma (> o)$.

Combining (2.4.13-17) we have

$$||S_{\lambda_n,k+1,m}(fg,t)-f(t)g(t)||_{C[a_2^*,b_2^*]} = O(\lambda_n^{-\tau/2}).$$

Since supp $fg \subset (a_2^*,b_2^*)$, by Lemma 2.4.2 $fg \in C_0(\alpha,k+1,a_2^*,b_2^*)$, whence $fg \in Liz(\alpha,k+1;a_2^*,b_2^*)$ (by Theorem 1.3.6). Noticing $g(t) = 1$ on $[a_2,b_2]$ it follows that $f \in Liz(\alpha,k+1;a_2,b_2)$, showing that (ii) holds.

This concludes the proof of Theorem 2.4.1.

2.5 SATURATION THEOREM FOR $S_{\lambda,k+1,m}(\cdot,t)$

In this section we prove a saturation theorem for the operators $S_{\lambda,k+1,m}(k \in \mathbb{N}^o)$ assuming that S_λ are regular exponential type operators.

THEOREM 2.5.1 : If S_λ are regular, $k,m \in \mathbb{N}^o$ and $f \in D_\psi(A,B)$, then in the following statements, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are true :

(i) $\quad \lambda_n^{k+1} \sup\limits_{t \in [a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = O(1);$

(ii) $\quad f^{(2k+1)} \in A.C.\,[a_2,b_2]$ and $f^{(2k+2)} \in L_\infty[a_2,b_2]$;

(iii) $\quad \lambda^{k+1}||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a_3,b_3]} = O(1)$;

(iv) $\quad \lambda_n^{k+1} \sup\limits_{t \in [a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = o(1)$;

(v) $\quad f \in C^{2k+2}[a_2,b_2]$ and $\sum\limits_{j=2}^{2k+2} Q(j,k+1,m,t)f^{(j)}(t) = o,$

$t \in [a_2, b_2]$ where $Q(j, k+1, m, t)$ are the polynomials occuring in (2.3.1);

(vi) $\quad \lambda^{k+1} || S_{\lambda, k+1, m}(f,t) - f(t) ||_{C[a_3, b_3]} = o(1).$

PROOF : First let us assume (i). Then, as in the proof of Theorem 2.4.1 it is clear that f is continuous on $[a_1, b_1]$. Moreover, in view of (i) $\Rightarrow$ (iii) of Theorem 2.4.1, it follows that $f^{(2k+1)}$ exists and is continuous on (a_1, b_1). Let $a_1 < a_1^* < a_2^* < a_2$ and $b_2 < b_2^* < b_1^* < b_1$. Then, we can choose a function f^* with supp $f^* \subset (a_1, b_1)$ such that $f^*(t) = f(t)$ on $[a_1^*, b_1^*]$ and that f^* is 2k+1 times continously differentiable on (a_1, b_1).

In view of Theorem 2.3.1, it is clear that

(2.5.1) $|| S_{\lambda_n, k+1, m}(f^*, t) - f^*(t) ||_{C[a_2^*, b_2^*]} = O(\lambda_n^{-(k+1)}).$

Let $C_o [a_1, b_1]$ denote the set of continuous functions q on (A, B) with supp $q \subset [a_1, b_1]$. Then $C_o[a_1, b_1]$ is a Banach space with the norm

(2.5.2) $||q|| = \max\limits_{t \in [a_1, b_1]} |q(t)|.$

Also, let $C_o^\infty (a_2^*, b_2^*)$ denote the space of infinitely differentiable functions g on (A, B) with supp $g \subset (a_2^*, b_2^*)$. Then there holds

(2.5.3) $\lambda^{k+1} \; | < S_{\lambda, k+1, m}(q, t) - q(t), \; g > | \leq M \; ||q||,$

for all λ sufficiently large, where M is a constant independent of λ . The proof of this assertion utilizes the identity

$$(2.5.4)\quad \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \binom{r+m}{i} = o, \; i \leq i \leq k,$$

which, however, shall be proved only in the last chapter of this thesis. Thus, assuming (2.5.4), on an application of Theorem 1.4.7 we get

$$\lambda^{k+1} < S_{\lambda,k+1,m}(q,t)-q(t),g >$$

$$= \lambda^{k+1} < q(u) , S^{*}_{\lambda,k+1,m}(g,u)-g(u) >$$

$$= \lambda^{k+1} < q(u), \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=1}^{r+m} \binom{r+m}{i} G_{i,k+1}(u,\lambda)+o(\lambda^{-(k+\cdots)})$$

(the o-term holding uniformly on $[a_1,b_1]$)

$$= \lambda^{k+1} < q(u), \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=k+1}^{r+m} \binom{r+m}{i} G_{i,k+1}(u,\lambda)+o(\lambda^{-(k+\cdots)})$$

But $G_{i,k+1}(u,\lambda) = O(\lambda^{-i}) = O(\lambda^{-(k+1)})$, for $i \geq k+1$, and therefore (2.5.3) follows.

Let $\{ f_\sigma \}$ be a sequence of 2k+2 times continuously differentiable functions on (A,B) with supp $f_\sigma \subset (a_1,b_1)$ and converging to f^* in the norm $||\cdot||$, defined by (2.5.2). Then, for any $g \in C_0^\infty(a_1,b_1)$ and each function f_σ , by Theorem 2.3.1 we have

$$\lim_{\lambda \to \infty} \lambda^{k+1} < S_{\lambda,k+1,m}(f_\sigma ,t)-f_\sigma(t),g(t) >$$

$$(2.5.5)\quad = < \sum_{j=2}^{2k+2} Q(j,k+1,m,t)\, f_\sigma^{(j)}(t),g(t) >$$

$$= < f_\sigma(t), \sum_{j=1}^{2k+2} Q^*(j,k+1,m,t)g^{(j)}(t) >,$$

where $Q^*_{2k+2}(D) = \sum_{j=1}^{2k+2} Q^*(j,k+1,m,t)D^j$ denotes the differential operator adjoint to $Q_{2k+2}(D) = \sum_{j=2}^{2k+2} Q(j,k+1,m,t)D^j$.

In view of (2.5.1), there exists a subsequence $\{ \lambda_{n_p}^{k+1} [S_{\lambda_{n_p},k+1,m}(f^*,t)-f^*(t)]\}_{p=1}^{\infty}$ converging to a function $h \in L_\infty[a_2^*,b_2^*]$ in the weak*- topology of the space $L_\infty[a_2^*,b_2^*]$. The dual of this space being $L_1[a_2^*,b_2^*]$, it follows, in particular, that

(2.5.6) $$\lim_{p\to\infty} \lambda_{n_p}^{k+1} < S_{\lambda_{n_p},k+1,m}(f^*,t)-f^*(t),g(t) >$$
$$= < h(t),g(t) >,$$

for every $g \in C_0^\infty(a_2^*,b_2^*)$.

By (2.5.3) we conclude that

(2.5.7) $$\lim_{p\to\infty} \lambda_{n_p}^{k+1} \,|< S_{\lambda_{n_p},k+1,m}(f^*-f_\sigma,t)-(f^*(t)-f_\sigma(t)),g(t)>$$
$$\le M||f^*-f_\sigma||.$$

Hence, by (2.5.7),(2.5.5) and (2.5.6)(in that order)

$$< f^*(t),Q^*_{2k+2}(D)g >$$

$$= \lim_{\sigma\to\infty} < f_\sigma(t), Q^*_{2k+2}(D)g>$$

$$= \lim_{\sigma\to\infty} [\lim_{p\to\infty} \lambda_{n_p}^{k+1} < S_{\lambda_{n_p},k+1,m}(f^*-f_\sigma,t)$$
$$- f^*(t)-f_\sigma(t),g(t)> + < f_\sigma(t), Q^*_{2k+2}(D)g >]$$

$$= \lim_{p\to\infty} \lambda_{n_p}^{k+1} < S_{\lambda_{n_p},k+1,m}(f^*,t)-f^*(t),g(t)>$$

$$= < h(t), g(t) >.$$

Thus,

$$(2.5.8) \qquad h(t) = Q_{2k+2}(D)\, f^*(t),$$

as generalized functions.

At this point we note that $Q(2k+2,k+1,m,t) \neq o$. To establish this, in view of Lemma 2.2.1 and Theorem 2.3.1, it is sufficient to show it for $m = o$ and for all $k = o,1,2,\ldots$. Equivalently, we have to prove that for $k = 1,2,\ldots$

$$\lim_{\lambda \to \infty} \lambda^k [\, I-(I-S_\lambda)^k\,]\,((u-t)^{2k},t) \neq o.$$

But this is obvious since this limit is easily seen to be equal to $(2k-1)!!\; p^k(t)$ which is non-zero.

Therefore, regarding (2.5.8) as a first order linear differential equation for $f^{*(2k+1)}$ with the non-homogeneous terms linearly depending on $f^{*(i)}$, $o \leq i \leq 2k$ and h with polynomial coefficients, as $f^{*(i)} \in C[\,a_2^*,b_2^*\,]$ $(o \leq i \leq 2k)$ and $h \in L_\infty[\,a_2^*,b_2^*\,]$ we conclude that $f^{*(2k+1)} \in A.C.\,[\,a_2^*,b_2^*\,]$ and therefore that $f^{*(2k+2)} \in L_\infty[\,a_2^*,b_2^*\,]$. Since $f \equiv f^*$ on $[\,a_2,b_2\,]$ it follows that $f^{(2k+1)} \in A.C.\,[\,a_2,b_2\,]$ and that $f^{(2k+2)} \in L_\infty[\,a_2,$ This completes the proof of the implication (i) $\Rightarrow$ (ii).

Now assuming (ii), it follows that $f^{(2k+1)} \in Lip_M(1;a_2,b_2)$ with $M = \|f^{(2k+2)}\|_{L_\infty[\,a_2,b_2\,]}$. Hence (iii) follows by Theorem 2.3.2.

To prove (iv) $\Rightarrow$ (v), assuming (iv) and proceeding in the manner of the proof of (i) $\Rightarrow$ (ii), we get

$$Q_{2k+2}(D)\, f^*(t) = o,$$

from which in view of the non-vanishing of $Q(2k+2,k+1,m,t)$, (v) is clear.

The proof of (v) $\Rightarrow$ (vi) follows from Theorem 2.3.1.

This completes the proof of the saturation theorem.

CHAPTER 3

SIMULTANEOUS APPROXIMATION WITH GENERALIZED MICCHELLI COMBINATIONS

3.1 INTRODUCTION

In the previous chapter we studied some approximation properties of the generalized Micchelli type combinations $S_{\lambda,k,m}$. The properties concerned the degree of approximation of f by $S_{\lambda,k,m}f$. In this chapter we investigate similar problems involved in approximating the derivatives $f^{(p)}$ by $S^{(p)}_{\lambda,k,m}f$. We first establish the basic pointwise convergence theorem in simultaneous approximation and then proceed to study the degree of this approximation, wherein we obtain direct, inverse and saturation theorems related with the rate of convergence of $S^{(p)}_{\lambda,k,m}f$.

3.2 BASIC RESULTS

In the following theorem, we show that the derivative $S^{(p)}_{\lambda,k,m}f$ is also an approximation process for $f^{(p)}$.

THEOREM 3.2.1 : Let $f \in D_\psi(A,B)$ and $k, p \in \mathbb{N}$. If $f^{(p)}$ exists at some point $t \in (A,B)$, then

$$(3.2.1) \qquad \lim_{\lambda \to \infty} S^{(p)}_{\lambda,k,m}(f,t) = f^{(p)}(t).$$

Further, if $f^{(p)}(t)$ exists and is continuous on $\langle a,b\rangle$, then (3.2.1) holds uniformly on $[a,b]$.

PROOF : To prove the theorem, it is sufficient to show that for each $r \in \mathbb{N}$,

$$(3.2.2) \qquad \lim_{\lambda \to \infty} D^p[\, S_\lambda^r(f,t)] = f^{(p)}(t) \quad (D \equiv \frac{d}{dt})$$

and that it holds uniformly in the uniformity case.

We can write

$$\begin{aligned}
D^p[\, S_\lambda^r(f,t)] &= \int_A^B W^{(p)}(\lambda,t,u_{r-1})\, S_\lambda^{r-1}(f,u_{r-1})du_{r-1} \\
&= \int_A^B W^{(p)}(\lambda,t,u_{r-1}) S_\lambda^{r-1}\Big(\sum_{i=o}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \epsilon(u,t)(u-t)^p, u_{r-1}\Big)du
\end{aligned}$$

(where $\epsilon(u,t) \to o$ as $u \to t$)

$$\begin{aligned}
&= \sum_{i=o}^{p} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(p)}(\lambda,t,u_{r-1})\, S_\lambda^{r-1}((u-t)^i, u_{r-1})du_{r-1} \\
&\quad + \int_A^B W^{(p)}(\lambda,t,u_{r-1}) S_\lambda^{r-1}(\epsilon(u,t)(u-t)^p, u_{r-1})du_{r-1} \\
&= I_1 + I_2, \text{ say.}
\end{aligned}$$

Let us estimate I_1 first. By Lemma 1.3.3 it is clear that

$$\begin{aligned}
I_1 &= \frac{f^{(p)}(t)}{p!} \int_A^B W^{(p)}(\lambda,t,u_{r-1}) S_\lambda^{r-1}(u^p, u_{r-1})du_{r-1} \\
&= \frac{f^{(p)}(t)}{p!} D^p\, [\, S_\lambda^r(u^p,t)]
\end{aligned}$$

Now applying Theorem 2.2.5 it follows that

$$I_1 \to f^{(p)}(t) \text{ as } \lambda \to \infty.$$

We estimate I_2 as follows :

Since $\epsilon(u,t) \to 0$ as $u \to t$, hence for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|\epsilon(u,t)| < \epsilon$ whenever $|u-t| \leq \delta$. If $\chi_\delta(u)$ is the characteristic function of the set $\{ u \in (A,B): |u-t| \leq \delta \}$ then,

$$|I_2| \leq \int_A^B |W^{(p)}(\lambda, t, u_{r-1})| \; [\, S_\lambda^{r-1}(|\epsilon(u,t)|\,|u-t|^p \chi_\delta(u), u_{r-1})$$
$$+ S_\lambda^{r-1}(|\epsilon(u,t)|\,|u-t|^p (1-\chi_\delta(u)), u_{r-1}) \,]\, du_{r-1}$$
$$= I_3 + I_4, \text{ say.}$$

Then, using Lemma 1.5.6, Schwarz inequality and Corollary 2.2.3 we have

$$I_3 \leq \epsilon \sum_{\substack{2i+j\leq p \\ i,j\geq 0}} \lambda^{i+j} \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p} \int_A^B W(\lambda, t, u_{r-1}) |u_{r-1}-t|^j .$$
$$\cdot [\, S_\lambda^{r-1}((u-t)^{2p}, u_{r-1}) \,]^{1/2} \; du_{r-1}$$

$$\leq \epsilon \sum_{\substack{2i+j\leq p \\ i,j\geq 0}} \lambda^{i+j} \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p} [(S_\lambda((u_{r-1}-t)^{2j}, t))^{1/2} .$$
$$\cdot (S_\lambda^r((u-t)^{2p}, t))^{1/2}]$$

$$\leq \epsilon . O(1).$$

Also, introducing the GTF Ψ, by Lemma 1.5.6 we get

$$I_4 \leq \sum_{\substack{2i+j\leq p \\ i,j\geq 0}} \lambda^{i+j} \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p} \int_A^B W(\lambda, t, u_{r-1}) |u_{r-1}-t|^j .$$
$$\cdot\, S_\lambda^{r-1}\left(\frac{M\,\Psi(u)(u-t)^{2m'}}{\delta^{2m'-p}}, u_{r-1}\right) du_{r-1},$$

where M is a constant and $m'(\in \mathbb{N}) > [p/2]$ is arbitrary. By an analysis similar to that of the proof of Lemma 2.4.2 we get $I_4 = O(\lambda^{(p-2m')/2})$.

Thus, $\quad I_4 = o(1)$.

Combining the above estimates, in view of the arbitrariness of $\epsilon > o$, the first assertion of the theorem follows.

To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $t \in [a,b]$ and also that the other estimates hold uniformly in $t \in [a,b]$.

Next, we establish an asymptotic formula.

<u>THEOREM</u> 3.2.2 : Let $f \in D_\psi(A,B)$ and $k,p \in \mathbb{N}$. If $f^{(2k+p)}$ exists at some point $t \in (A,B)$, then

$$(3.2.2) \qquad \lim_{\lambda \to \infty} \lambda^k \left[S^{(p)}_{\lambda,k,m}(f,t) - f^{(p)}(t) \right] = \sum_{j=p}^{2k+p} Q(j,k,m,p,t) f^{(j)}(t)$$

and

$$(3.2.3) \qquad \lim_{\lambda \to \infty} \lambda^k \left[S^{(p)}_{\lambda,k+1,m}(f,t) - f^{(p)}(t) \right] = o,$$

where $Q(j,k,m,p,t)$ are certain polynomials in t.

Further, if $f^{(2k+p)}$ exists and is continous on $\langle a,b \rangle$ then (3.2.2-3) hold uniformly on $[a,b]$.

<u>PROOF</u> : Using a partial Taylor expansion of f, we have

$$S^{(p)}_{\lambda,k,m}(f,t) = \sum_{r=1}^{k} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m}{k-r} \int_A^B W^{(p)}(\lambda,t,u_{r+m-1}).$$

$$\cdot\; S_\lambda^{r+m-1} \Big(\sum_{i=0}^{2k+p} \frac{f^{(i)}(t)}{i!}(u-t)^i$$

$$+ \epsilon(u,t)(u-t)^{2k+p}, u_{r+m-1} \Big) du_{r+m-1},$$

Now, with $D \equiv \frac{d}{dt}$, by Lemma 1.3.3 and Theorem 2.3.1 we have

$$I_1 = \sum_{i=p}^{2k+p} \frac{f^{(i)}(t)}{i!} \sum_{\ell=o}^{i} \binom{i}{\ell} (-1)^{i-\ell} t^{i-\ell} S^{(p)}_{\lambda,k,m}(u^{\ell}, t)$$

$$= \sum_{i=p}^{2k+p} \frac{f^{(i)}(t)}{i!} \sum_{\ell=o}^{i} \binom{i}{\ell} (-1)^{i-\ell} t^{i-\ell} [D^p t^{\ell}$$

$$+ \lambda^{-k} \{ \sum_{j=2}^{2k} D^p [Q(j,k,m,t) D^j t^{\ell}] + o(1) \}]$$

$$= \sum_{i=p}^{2k+p} \frac{f^{(i)}(t)}{i!} p! \sum_{\ell=o}^{i} \binom{i}{\ell} (-1)^{i-\ell} \binom{\ell}{p} t^{i-p}$$

$$+ \lambda^{-k} \sum_{j=p}^{2k+p} Q(j,k,m,p,t) f^{(j)}(t) + o(\lambda^{-k}),$$

using the identities

$$\sum_{\ell=o}^{i} (-1)^{\ell} \binom{i}{\ell} \binom{\ell}{p} = \begin{cases} o, & i > p \\ (-1)^p, & i = p \end{cases} .$$

Next, with $\chi_\delta(u)$ defined as in the proof of Theorem 3.2.1, by Lemma 1.5.6 we have

$$|I_2|$$

$$\leq \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} \sum_{\substack{2j+j\leq p \\ i,j\geq o}} \lambda^{i+j} \frac{|q^{[p]}_{ij}(t)|}{(p(t))^p} \int_A^B W(\lambda,t,u_{r+m-1}) |u_{r+m-1}$$

$$\cdot [S_\lambda^{r+m-1}(|\epsilon(u,t)| |u-t|^{2k+p} \chi_\delta(u), u_{r+m-1})$$

$$+ S_\lambda^{r+m-1}(|\epsilon(u,t)| |u-t|^{2k+p}(1-\chi_\delta(u)), u_{r+m-1})] du_{r+m-1}$$

$$= I_3 + I_4, \text{ say.}$$

Then, as in the proof of Theorem 3.2.1 we can easily show that

$$I_3 \leq \epsilon\; O(\lambda^{-k}), \text{ and}$$

$$I_5 = o(\lambda^{-k}).$$

Combining these estimates and using the arbitrariness of $\epsilon > o$, we get (3.2.2). The assertion (3.2.3) can be proved similarly usi The uniformity assertion follows as in the proof of Theorem 2.3

Next we prove an analogue of Theorem 2.3.2.

<u>THEOREM</u> 3.2.3 : Let $p \leq q \leq 2k+p$, $f \in D_\Psi(A,B)$ and $f^{(q)}$ exist and be continuous on $\langle a,b\rangle$. Then

$$(3.2.4)\quad ||S^{(p)}_{\lambda,k,m}(f,t)-f^{(p)}(t)||_{C[a,b]} \leq \max\{ C\,\lambda^{-(q-p)/2}\,\omega(f^{(q)}, \lambda^{-1/2}),\ C'\lambda^{-k}\},$$

where $C = C(k,m,p)$ and $C' = C'(k,m,p,f)$.

<u>PROOF</u> : We can write

$$(3.2.5)\quad f(u) = \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(q)}(\xi)-f^{(q)}(t)}{q!}(u-t)^q + h(u,t)\,\chi(u),\ u \in (A,B)$$

where ξ lies between u and t and $\chi(u)$ is the characteristic function of the set $(A,B)\setminus\langle a,b\rangle$. The function $h(u,t)$ for $t \in [a,b]$ is bounded by $M\Psi(u)(u-t)^q$ for some constant M. Operating on this equality by $S^{(p)}_{\lambda,k,m}$ and breaking right hand side into three parts I_1, I_2 and I_3, say, corresponding to the three terms on the right hand side of (3.2.5), as in the

proof of the previous theorem it can be easily shown that

$$I_3 = o(\lambda^{-k}),$$

uniformly in $t \in [a,b]$.

By Theorem 3.2.2, $I_1 = f^{(p)}(t) + O(\lambda^{-k})$ uniformly in $t \in [a,b]$. Finally,

$$|I_2| \leq \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} \sum_{\substack{2i+j\leq p \\ i,j\geq o}} \lambda^{i+j} \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p} \int_A^B W(\lambda, t, u_{r+m-1})$$

$$\cdot |u_{r+m-1}-t|^j S_\lambda^{r+m-1}\left(\frac{|f^{(q)}(\xi)-f^{(q)}(t)|}{q!}|u-t|^q, u_{r+m-1}\right) du_{r+m-1}$$

$$\leq \frac{\omega(f^{(q)}, \lambda^{-1/2})}{q!} \sum_{r=1}^{k} \frac{1}{m\beta(m,r)} \binom{k+m}{k-r} \sum_{\substack{2i+j\leq p \\ i,j\geq o}} \lambda^{i+j} \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p}$$

$$\cdot \int_A^B W(\lambda, t, u_{r+m-1}) |u_{r+m-1}-t|^j [S_\lambda^{r+m-1}(|u-t|^q, u_{r+m-1})$$

$$+ \lambda^{1/2} S_\lambda^{r+m-1}(|u-t|^{q+1}, u_{r+m-1})] \; du_{r+m-1}$$

$$= \omega(f^{(q)}, \lambda^{-1/2}) \sum_{r=1}^{k} \sum_{\substack{2i+j\leq p \\ i,j\geq o}} O\left(\lambda^{\frac{(2i+j-q)}{2}}\right)$$

$$= \omega(f^{(q)}, \lambda^{-1/2}) \; O(\lambda^{-(q-p)/2}),$$

where the O-term holds uniformly in $t \in [a,b]$. Here an involved use of Schwarz inequality and the Corollary 2.2.3 has been made.

The theorem follows from these estimates of I_1, I_2 and I_3.

This completes our discussion of direct theorems in the simultaneous approximation $S_{\lambda,k,m}^{(p)} f \to f^{(p)}$.

3.3 INVERSE THEOREM FOR $S_{\lambda,k+1,m}^{(p)}(.,t)$

In this section we shall be concerned with the proof of the following

THEOREM 3.3.1 : Let $o < \alpha < 2$ and $f \in D_\psi(A,B)$. Then, in the following statements, the implications (i) $\Rightarrow$(ii)$\Leftrightarrow$ (iii) $\Rightarrow$ (iv) hold.

(i) $f^{(p)}$ exists on $[a_1, b_1]$ and

$$\sup_{t \in [a_1,b_1]} |S_{\lambda_n,k+1,m}^{(p)}(f,t) - f^{(p)}(t)| = O(\lambda_n^{-\alpha(k+1)/2});$$

(ii) $f^{(p)} \in Liz(\alpha, k+1; a_2, b_2)$;

(iii) (a) For $m' < \alpha(k+1) < m'+1, m' = o,1,2,\ldots,2k+1$: $f^{(m'+p)}$ exists and $\in Lip(\alpha(k+1)-m'; a_2, b_2)$, and

(b) For $\alpha(k+1) = m'+1$, $m' = o,1,2,\ldots,2k$: $f^{(m'+p)}$ exists and $\in Lip^*(1; a_2, b_2)$;

(iv) $$||S_{\lambda,k+1,m}^{(p)}(f,t) - f^{(p)}(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}).$$

Before giving a proof of this inverse theorem we first define an intermediate space $C_o^p(\alpha, k; a', b')$ as follows :

Let $[a,b]$ be a fixed subinterval of (A,B) and let $[a',b'] \subset (a,b)$. Let us denote $G^{(p)} = \{g : g \in C_o^{2k+2}$,

supp $g \subset [a', b']\}$. For p times continuously differentiable functions f with supp $f \subset [a', b']$ we define

$$(3.3.1)\ K_p(\xi, f) = \inf_{g \in G^{(p)}} \{ \|f^{(p)} - g^{(p)}\|_{C[a',b']} + \xi (\|g\|_{C[a',b']} + \|g^{(2k+p+2)}\|_{C[a',b']}) \},$$

where $o < \xi \leq 1$. For $o < \alpha < 2$, we define $C_o^p(\alpha, k+1; a', b')$ as the class of all p times continuously differentiable functions f with supp $f \subset [a', b']$ such that the functional

$$\|f\|_{\alpha,p} \equiv \sup_{o < \xi \leq 1} \xi^{-\alpha/2} K_p(\xi, f) < M,$$

for some constant $M > o$.

We prove some auxiliary results :

<u>LEMMA</u> 3.3.2 : Let $a < a' < a'' < b'' < b' < b$. If $f^{(p)} \in C_o$ with supp $f \subset [a'', b'']$ and $\|S^{(p)}_{\lambda_n, k+1, m}(f, t) - f^{(p)}(t)\|_{C[a,b]} \leq M \lambda_n^{-\alpha(k+1)/2}$, then

$$(3.3.2)\ K_p(\xi, f) \leq M_o [\lambda^{-\alpha(k+1)/2} + \lambda^{k+1} \xi K_p(\lambda^{-(k+1)}, f)].$$

Consequently, $K_p(\xi, f) \leq M' \xi^{\alpha/2}$ for some constant M' i.e., $f \in C_o^p(\alpha, k+1; a', b')$.

<u>PROOF</u> : As in the proof of Lemma 2.4.2, it is enough to show that

$$(3.3.3)\ K_p(\xi, f) \leq M_o [\lambda_n^{-\alpha(k+1)/2} + \lambda_n^{k+1} \xi K_p(\lambda_n^{-(k+1)}, f)],$$

for all n sufficiently large.

Since supp $f \subset [a'', b'']$, as in the proof of Lemma 2.4.2, using Theorem 3.2.2 we can find a function $h_n \in G^{(p)}$ such that for $i = p$ and $2k+p+2$, there holds

$$||h_n^{(i)}(t) - S^{(i)}_{\lambda_n,k+1,m}(f,t)||_{C[a,b]} \leq M_1 \lambda_n^{-(k+1)},$$

for all n sufficiently large.

Therefore, for some constant M_1 we have

$$K_p(\xi, f) \leq 3M_1 \lambda_n^{-(k+1)} + ||S^{(p)}_{\lambda_n,k+1,m}(f,t) - f^{(p)}(t)||_{C[a',b']}$$

$$+ \xi[||S^{(p)}_{\lambda_n,k+1,m}(f,t)||_{C[a',b']} + ||S^{(2k+p+2)}_{\lambda_n,k+1,m}(f,t)||_{C[a}$$

Hence it is enough to show that there exists an M_2, such that, for each $g \in G^{(p)}$

$$||S^{(2k+p+2)}_{\lambda,k+1,m}(f,t)||_{C[a',b']} \leq M_2 \lambda^{k+1} [\,||f^{(p)} - g^{(p)}||_{C[a',b']}$$

$$+ \lambda^{-(k+1)}||g^{(2k+p+2)}||_{C[a',b']}]\,.$$

Now,

$$||S^{(2k+p+2)}_{\lambda,k+1,m}(f,t)||_{C[a',b']}$$

$$\leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1} \,||\, \int_A^B W^{(2k+p+2)}(\lambda, t, u_{r+m-1})\,.$$

$$\cdot\, S_\lambda^{r+m-1}((f-g)(u), u_{r+m-1})du_{r+m-1}||_{C[a',b']}$$

$$+ \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}||\int_A^B W^{(2k+p+2)}(\lambda, t, u_{r+m-1})\,. \tag{3.3.4}$$

$$\cdot\, S_\lambda^{r+m-1}(g(u), u_{r+m-1})du_{r+m-1}||_{C[a',b']}$$

$$= I_1 + I_2, \text{ say.}$$

But, by Lemmas 1.3.3, 1.5.6 Schwarz inequality and Corollary 2.2.3 we have

$$\left\|\int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1})\, S_\lambda^{r+m-1}((f-g)(u),u_{r+m-1})\,du_{r+m-1}\right\|_{C[a',b']}$$

$$= \left\|\int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1})\,S_\lambda^{r+m-1}\left(\sum_{\ell=0}^{p-1}\frac{(f-g)^{(\ell)}(t)}{\ell!}(u-t)^\ell + \frac{(f-g)^{(p)}(\xi)}{p!}(u-t)^p,u_{r+m-1}\right)du_{r+m-1}\right\|_{C[a',b']}$$

(ξ lying between u and t)

$$= \left\|\int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1})\,S_\lambda^{r+m-1}\left(\frac{(f-g)^{(p)}(\xi)}{p!}(u-t)^p,u_{r+m-1}\right)\cdot du_{r+m-1}\right\|_{C[a',b']}$$

$$\leq \frac{\|f^{(p)}-g^{(p)}\|_{C[a',b']}}{p!}\left\|\sum_{\substack{2i+j\leq 2k+p+2\\ i,j\geq 0}}\lambda^{i+j}\,\frac{|q_{ij}^{[2k+p+2]}(t)|}{(p(t))^{2k+p+2}}\cdot\int_A^B W(\lambda,t,u_{r+m-1})\cdot |u_{r+m-1}-t|^j\,S_\lambda^{r+m-1}(|u-t|^p,u_{r+m-1})\,du_{r+m-1}\right\|_{C[a',b']}$$

(supp f $\cup$ supp g $\subset$ [a', b'])

$$\leq M_2\,\lambda^{k+1}\,\|f^{(p)}-g^{(p)}\|_{C[a',b']}\,.$$

Hence $I_1 \leq M_3 \lambda^{k+1} \; ||f^{(p)}-g^{(p)}||_{C[a',b']}$.

Now we shall estimate I_2.

By Lemmas 1.3.3, 1.5.6, Schwarz inequality and Corollary 2.2.3,

$$|| \int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1}) S_\lambda^{r+m-1}(g(u),u_{r+m-1})du_{r+m-1}||_{C[a',b']}$$

$$= || \int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1}) S_\lambda^{r+m-1}\Big(\sum_{\ell=0}^{2k+p+1} \frac{g^{(\ell)}(t)}{\ell!}(u-t)^\ell$$

$$+ \frac{g^{(2k+p+2)}(\xi)}{(2k+p+2)!}(u-t)^{2k+p+2}, u_{r+m-1}\Big)du_{r+m-1}||_{C[a',b']}$$

$$= ||\int_A^B W^{(2k+p+2)}(\lambda,t,u_{r+m-1}) S_\lambda^{r+m-1}\Big(\frac{g^{(2k+p+2)}(\xi)}{(2k+p+2)!}(u-t)^{2k+p+2}, u_{r+m-1}\Big)\cdot$$

$$\cdot du_{r+m-1}||_{C[a',b']}$$

$$\leq \frac{||g^{(2k+p+2)}||_{C[a',b']}}{(2k+p+2)!} \; || \sum_{\substack{2i+j\leq 2k+p+2 \\ i,j\geq 0}} \lambda^{i+j} \; \frac{|q_{ij}^{[2k+p+2]}(t)|}{(p(t))^{2k+p+2}} \cdot$$

$$\cdot \int_A^B W(\lambda,t,u_{r+m-1})|u_{r+m-1}-t|^j \; S_\lambda^{r+m-1}(|u-t|^{2k+p+2}, u_{r+m-1})du_{r+m-1}||_{C[a',b']}$$

$$\leq M_4||g^{(2k+p+2)}||_{C[a',b']} .$$

Therefore, $I_2 \leq M_5 \; ||g^{(2k+p+2)}||_{C[a',b']}$.

Substituting the estimates of I_1 and I_2 into (3.3.4), (3.3.3) follows. This completes the proof of Lemma 3.3.2.

<u>LEMMA</u> 3.3.3 : Let $a < a' < a'' < b'' < b' < b$. If $f^{(p)} \in C_o$ with supp $f \subset [a'',b'']$ and $f \in C_o^p(\alpha, k+1; a', b')$ then

(3.3.5) $$||S^{(p)}_{\lambda,k+1,m}(f,t) - f^{(p)}(t)||_{C[a,b]} \leq M\lambda^{-\alpha(k+1)/2}.$$

<u>PROOF</u> : For $g \in G^{(p)}$, we have

$$||S^{(p)}_{\lambda,k+1,m}(f-g,t)||_{C[a,b]}$$

$$\leq ||S^{(p)}_{\lambda,k+1,m}(f-g,t)||_{C[a,b]} + ||S^{(p)}_{\lambda,k+1,m}(g,t) - f^{(p)}(t)||_{C[a,b]}$$

$= I_1 + I_2$, say.

It is easily seen that

$$I_1 \leq M_1||f^{(p)} - g^{(p)}||_{C[a',b']} \quad \text{(since supp } (f-g) \subset [a',b']).$$

I_2 can be estimated as follows:

$$I_2 \leq ||g^{(p)} - f^{(p)}||_{C[a',b']} + M_2\lambda^{-(k+1)} \sum_{j=p}^{2k+p+2} ||g^{(j)}||_{C[a',b']}$$

(by Theorem 3.2.2)

$$\leq ||g^{(p)} - f^{(p)}||_{C[a',b']} + M_3\lambda^{-(k+1)}(||g||_{C[a',b']}$$

$$+ ||g^{(2k+p+2)}||_{C[a',b']}).$$

Thus,

$$||S^{(p)}_{\lambda,k+1,m}(f,t)-f^{(p)}(t)||_{C[a,b]} \leq M_4 K_p(\lambda^{-(k+1)},f)$$
$$\leq M_5 \lambda^{-\alpha(k+1)/2},$$

since $f \in C_0^p(\alpha,k+1;a',b')$.

This proves the required result.

LEMMA 3.3.4 : Let $a < a' < a'' < b'' < b' < b$. If $f^{(p)} \in C_0$ with supp $f \subset [a'',b'']$ then $f \in C_0^p(\alpha,k+1;a',b')$ iff $f^{(p)} \in Liz(\alpha,k+1;a',b')$.

PROOF : Let $|\delta| < h$ and $g \in G^{(p)}$. Then, if $f \in C_0^p(\alpha,k+1;a',b')$

$$|\Delta_\delta^{2k+2} f^{(p)}(t)| \leq |\Delta_\delta^{2k+2}(f^{(p)}(t)-g^{(p)}(t))| + |\Delta_\delta^{2k+2} g^{(p)}(t)|$$

$$\leq 2^{2k+2}\,||f^{(p)}-g^{(p)}||_{C[a',b']} + \delta^{2k+2}||g^{(2k+p+2)}||_{C[a',b']}$$

$$\leq 2^{2k+2}\,K_p(\delta^{2k+2},f)$$

$$\leq 2^{2k+2} M\,\delta^{\alpha(k+1)}.$$

It follows that

$$\omega_{2k+2}(f^{(p)},h;a',b') = \sup_{|\delta|\leq h} |\Delta_\delta^{2k+2} f^{(p)}(t)| \leq M' h^{\alpha(k+1)}$$

i.e., $f^{(p)} \in Liz(\alpha,k+1;a',b')$.

Conversely, let $f^{(p)} \in Liz(\alpha,k+1;a',b')$.

1st Case : When p is even, say $p = 2j$.

Define $g_o \in G^{(p)}$ by

$$g_o(x) = \frac{1}{\binom{2k+2j+2}{k+j+1}\eta^{2k+2j+2}} \int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2} [(-1)^{k+j}\cdot$$

$$\cdot \Delta^{2k+2j+2}_{\sum_{\nu=1}^{2k+2j+2} u_\nu} f(x) + \binom{2k+2j+2}{k+j+1} f(x)]\, du_1 du_2 \ldots du_{2k+2j+2},$$

where $(k+j+1)^2\eta < \min(a''-a', b'-b'')$ and Δ_h^r is the r-th symmetric difference operator.

Then,

$$(-1)^{k+j}\binom{2k+2j+2}{k+j+1}\eta^{2k+2j+2} \quad g_o(x)$$

$$= \int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}[\sum_{i=o}^{2k+2j+2}(-1)^i\binom{2k+2j+2}{k+j+1} f(x+(k+j+1-i)\cdot$$

$$\cdot\sum_{\nu=1}^{2k+2j+2} u_\nu) + (-1)^{k+j}\binom{2k+2j+2}{k+j+1} f(x)]\, du_1 du_2 \ldots du_{2k+2j+2}$$

$$= \int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}[\sum_{\substack{i=o\\ i\neq k+j+1}}^{2k+2j+2}(-1)^i\binom{2k+2j+2}{i} f(x+(k+j+1-i)\cdot$$

$$\cdot\sum_{\nu=1}^{2k+2j+2} u_\nu)]\, du_1 du_2 \ldots, du_{2k+2j+2}$$

$$= \int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}[\sum_{i=o}^{k+j}(-1)^i\binom{2k+2j+2}{k+j+1} f(x+(k+j+1-i)\cdot$$

$$\cdot\sum_{\nu=1}^{2k+2j+2} u_\nu) + \sum_{i=o}^{k+j}(-1)^i\binom{2k+2j+2}{i} f(x-(k+j+1-i)\sum_{\nu=1}^{2k+2j+2} u_\nu)]\cdot$$

$$\cdot\, du_1 du_2 \ldots du_{2k+2j+2}$$

$$\doteq \int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\sum_{i=0}^{k+j}(-1)^i\binom{2k+2j+2}{k+j+1}\{f(x+(k+j+1-i)\cdot$$

$$\cdot\sum_{\nu=1}^{2k+2j+2}u_\nu)+f(x-(k+j+1-i)\sum_{\nu=1}^{2k+2j+2}u_\nu)\}\,du_1du_2\ldots du_{2k+2j+2}.$$

Since

$$\frac{d^{2k+2j+2}}{dx^{2k+2j+2}}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}[\,f(x+\sum_{\nu=1}^{2k+2j+2}u_\nu)$$

$$+f(x-\sum_{\nu=1}^{2k+2j+2}u_\nu)\,]\,du_1du_2\ldots du_{2k+2j+2}$$

$$=\;2\;\Delta_\eta^{-2k+2j+2}\,f(x),$$

and

$$\omega_{2k+2j+2}(f,|k+j+1-i|\,\eta)\leq|k+j+1-i|^{2k+2j+2}\;\omega_{2k+2j+2}(f,\eta)$$

$$\leq M_1\,\omega_{2k+2j+2}(f,\eta),$$

therefore,

$$||g_o^{(2k+2j+2)}||_{C[a',b']}$$

$$=\frac{\eta^{-(2k+2j+2)}}{\binom{2k+2j+2}{k+j+1}}\,||\sum_{i=0}^{k+j}(-1)^i\binom{2k+2j+2}{i}\,2\,\Delta^{-2k+2j+2}_{(k+j+1-i)\eta}\,f(x)||_{C[a',b']}$$

$$\leq\frac{\eta^{-(2k+2j+2)}}{\binom{2k+2j+2}{k+j+1}}\;2\sum_{i=0}^{k+j}M_1\binom{2k+2j+2}{i}\,\omega_{2k+2j+2}(f,\eta\,;a',b')$$

$$\leq M_1 \eta^{-(2k+2j+2)} \eta^{2j} \omega_{2k+2}(f^{(2j)}, \eta ; a', b')$$

$$\leq M_2 \eta^{-(2k+2)+\alpha(k+1)} .$$

Next,

$$g_o^{(2j)}(x) = \frac{1}{\binom{2k+2j+2}{k+j+1} \eta^{2k+2j+2}} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} [(-1)^{k+j} .$$

$$\cdot \Delta^{-2k+2}_{\sum_{\nu=1}^{2k+2j+2} u_\nu} (D^{2j} \Delta^{-2j}_{\sum_{\nu=1}^{2k+2j+2} u_\nu} f(x)) + \binom{2k+2j+2}{k+j+1} f^{(2j)}(x)] .$$

$$\cdot du_1 du_2 \ldots, du_{2k+2j+2} ,$$

where $D \equiv \frac{d}{dx}$.

Hence,

$$g_o^{(2j)}(x) - f^{(2j)}(x) = \frac{1}{\binom{2k+2j+2}{k+j+1} \eta^{2k+2j+2}} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} [(-1)^{k+j} .$$

$$\cdot \sum_{i=o}^{2j} (-1)^i \binom{2j}{i} \Delta^{-2k+2}_{\sum_{\nu=1}^{2k+2j+2} u_\nu} f^{(2j)}(x_{-j} + i \sum_{\nu=1}^{2k+2j+2} u_\nu)] .$$

$$\cdot du_1 du_2 \ldots, du_{2k+2j+2} .$$

So,

$$||g_o^{(2j)}-f^{(2j)}||_{C[a',b']} \leq \frac{1}{\binom{2k+2j+2}{k+j+1}n^{2k+2j+2}} \cdot$$

$$\cdot[\omega_{2k+2}(f^{(2j)},(k+j+1)n\ ) \cdot$$

$$\cdot(\sum_{i=o}^{2j}\binom{2j}{i})\ n^{2k+2j+2}]$$

$$\leq M_3\ \omega_{2k+2}(f^{(2j)},n\ ;a',b')$$

$$\leq M_4\ n^{\alpha(k+1)}.$$

<u>2nd case</u> : When p is odd. Defining $g_o \in G^{(p)}$ as

$$g_o(x) = \frac{1}{\binom{2k+p+2}{k+[\frac{p}{2}]+1}n^{2k+2j+2}} \int_{-n/2}^{n/2}\int_{-n/2}^{n/2}\cdots\int_{-n/2}^{n/2}[\ (-1)^{k+[\ p/2]}.$$

$$\cdot\ \Delta^{-2k+p+2}_{\sum_{\nu=1}^{2k+p+2}u_\nu}\ f(x-1/2) + \binom{2k+p+2}{k+[\frac{p}{2}]+1}f(x)]\ du_1du_2\cdots du_{2k+p+2},$$

and proceeding as in the first case, we obtain

$$||g_o^{(2k+p+2)}||_{C[a',b']} \leq M_5\ n^{-(2k+2)+\alpha(k+1)}\ ,$$

and

$$||g_o^{(p)}-f^{(p)}||_{C[a',b']} \leq M_6\ n^{\alpha(k+1)}\ .$$

Thus, combining the two cases we have

$$\|g_o^{(2k+p+2)}\|_{C[a',b']} \leq M_7\, \eta^{-(2k+2)+\alpha(k+1)} ,$$

and

$$\|g_o^{(p)}-f^{(p)}\|_{C[a',b']} \leq M_8\, \eta^{\alpha(k+1)}.$$

From these estimates it is clear that

$$K_p(\eta^{2k+2},f) \leq M_9\, \eta^{\alpha(k+1)} .$$

Hence $f \in C_0^p(\alpha,k+1;a',b')$.

This completes the proof.

PROOF OF THEOREM 3.3.1 : Following the proof of Theorem 2.4.1 we have to show the implications (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii). We also observe that (i) implies that $f^{(p)} \in C[a_1,b_1]$. To prove the implication (ii) $\Rightarrow$ (iv) let a',a'',b',b'' and g be as in the proof of (ii) $\Rightarrow$ (iv) of Theorem 2.4.1. Then, as also $(fg)^{(p)} \in \mathrm{Liz}(\alpha,k+1;a_2,b_2)$ and $\mathrm{supp}\, fg \subset [a',b']$, by Lemmas 3.3.3-4 we have

$$\|S^{(p)}_{\lambda,k+1,m}(fg,t)-(fg)^{(p)}(t)\|_{C[a_2,b_2]} = O(\lambda^{-\alpha(k+1)/2}).$$

Consequently, since $g(t) = 1$ on $[a'',b'']$ and $[a_3,b_3] \subset (a'',b'')$,

$$\|S^{(p)}_{\lambda,k+1,m}(fg,t)-f^{(p)}(t)\|_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}). \tag{3.3.6}$$

To show that

$$\|S^{(p)}_{\lambda,k+1,m}(fg,t)-S^{(p)}_{\lambda,k+1,m}(f,t)\|_{C[a_3,b_3]} = o(\lambda^{-(k+1)}), \tag{3.3.7}$$

let $\delta \le \min(a_3-a'', b''-b_3)$ and $\chi_1(u)$ denote the characteristic function of the set $(A,B)\setminus[a'',b'']$. Then, by Lemma 1.5.6, Schwarz inequality and Corollary 2.2.3 we have

$$||S^{(p)}_{\lambda,k+1,m}(fg-f,t)||_{C[a_3,b_3]}$$

$$= ||S^{(p)}_{\lambda,k+1,m}((fg-f)\,\chi_1(u),t)||_{C[a_3,b_3]}$$

$$\le \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)}\binom{k+m+1}{k-r+1}||\sum_{\substack{2i+j\le p\\ i,j\ge o}} \lambda^{i+j}\frac{|q^{[p]}_{ij}(t)|}{(p(t))^{p}}\int_A^B W(\lambda,t,u_{r+m-1}).$$

$$\cdot|u_{r+m-1}-t|^{j}\, S^{r+m-1}_{\lambda}\left(\frac{M\,\Psi(u)(u-t)^{2m''}}{\delta^{2m''}},u_{r+m-1}\right)du_{r+m-1}||_{C[a_3,b_3]}$$

$$(m''\ (\in \mathbb{N}) > [\tfrac{2k+p+2}{2}] \text{ being arbitrary})$$

$$= O(\lambda^{(p-2m'')/2}),$$

and (3.3.7) follows due to the arbitrariness of m''.

Hence, by (3.3.6) and (3.3.7) we have (iv).

To prove the implication (i) $\Rightarrow$ (ii), writing $\tau = \alpha(k+1)$ we first consider the case $o < \tau \le 1$. Let a', a'', b', b'' and g be as in the proof of (i) $\Rightarrow$ (ii) of Theorem 2.4.1.

For $t \in [a',b']$, with $D \equiv \frac{d}{dt}$, we have

$$S^{(p)}_{\lambda_n,k+1,m}(fg,t)-(fg)^{(p)}(t)$$

$$= D^{p}[S_{\lambda_n,k+1,m}((fg)(u)-(fg)(t),t)]$$

$$(3.3.8)\quad = D^{p}[S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t)),t)]$$

$$+ D^{p}[S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)]$$

$$= I_1+I_2, \text{ say.}$$

We shall estimate I_1 first. By Leibniz theorem

$$I_1 = \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \frac{\partial^p}{\partial t^p} \int_A^B W(\lambda_n, t, u_{r+m-1}) .$$

$$. S_{\lambda_n}^{r+m-1}(f(u)(g(u)-g(t)),\ u_{r+m-1}) du_{r+m-1}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p} \binom{p}{i} \int_A^B W^{(i)}(\lambda_n, t, u_{r+m-1}) .$$

$$\cdot \frac{\partial^{p-i}}{\partial t^{p-i}} [S_{\lambda_n}^{r+m-1}(f(u)(g(u)-g(t)), u_{r+m-1})] du_{r+m-1}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p-1} \binom{p}{i} \int_A^B W^{(i)}(\lambda_n, t, u_{r+m-1}) .$$

(3.3.9)

$$\cdot \frac{\partial^{p-i}}{\partial t^{p-i}} [S_{\lambda_n}^{r+m-1} (f(u)(g(u)-g(t)), u_{r+m-1})] du_{r+m-1}$$

$$+ \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}) .$$

$$\cdot S_{\lambda_n}^{r+m-1} (f(u)(g(u)-g(t)), u_{r+m-1}) du_{r+m-1}$$

$$= I_3 + I_4, \text{ say.}$$

Now

$$I_3 = -\sum_{i=o}^{p-1} \binom{p}{i} g^{(p-i)}(t)\ S^{(i)}_{\lambda_n, k+1, m}(f, t)$$

(3.3.10)

$$= -\sum_{i=o}^{p-1} \binom{p}{i} g^{(p-i)}(t) f^{(i)}(t) + O(\lambda_n^{-\tau/2})$$

by Theorem 3.2.3, uniformly in $t \in [a', b']$.

To estimate I_4, with the hypothesis on f and g we can write

$$f(u) = \sum_{i=o}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + o(u-t)^p$$

and

$$g(u)-g(t) = \sum_{i=o}^{p+1} \frac{f^{(i)}(t)}{i!}(u-t)^i + o(u-t)^{p+1} .$$

Using these Taylor expansions of f and g in I_4, by Schwarz inequality and Corollary 2.2.3, it is easily seen that

$$I_4 = \sum_{i=1}^{p} \frac{g^{(i)}(t)}{i!} \frac{f^{(p-i)}(t)}{(p-i)!} p! + O(\lambda_n^{-1/2})$$

(3.3.11)
$$= \sum_{i=1}^{p} \binom{p}{i} g^{(i)}(t) f^{(p-i)}(t) + O(\lambda_n^{-\tau/2})$$
(uniformly in $t \in [a', b']$).

Combining these estimates of I_3 and I_4, we see that $I_1 = O(\lambda_n^{-\tau/2})$.

To estimate I_2, by Leibniz theorem

$$I_2 = \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p} \binom{p}{i} \int_A^B W^{(i)}(\lambda_n, t, u_{r+m-1}) .$$

$$\cdot \frac{\partial^{p-i}}{\partial t^{p-i}} S_{\lambda_n}^{r+m-1} (g(t)(f(u)-f(t)), u_{r+m-1}) du_{r+m-1}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p} \binom{p}{i} g^{(p-i)}(t) \Big[\int_A^B W^{(i)}(\lambda_n, t, u_{r+m-1}) .$$

(3.3.12)
$$\cdot S_{\lambda_n}^{r+m-1}(f(u), u_{r+m-1}) du_{r+m-1} \Big] - (fg)^{(p)}(t)$$

$$= \sum_{i=o}^{p} \binom{p}{i} g^{(p-i)}(t) S_{\lambda_n, k+1, m}^{(i)}(f,t) - (fg)^{(p)}(t)$$

$$= \sum_{i=o}^{p} \binom{p}{i} g^{(p-i)}(t) f^{(i)}(t) - (fg)^{(p)}(t) + O(\lambda_n^{-\tau/2})$$

(by Theorem 3.2.3 and the hypothesis that (i) holds)

$$= O(\lambda_n^{-\tau/2}) \quad \text{(uniformly in } t \in [a',b'] \text{).}$$

Consequently,

$$\| S^{(p)}_{\lambda_n,k+1,m}(fg,t) - (fg)^{(p)}(t) \|_{C[a',b']} = O(\lambda_n^{-\tau/2}).$$

Hence, by Lemmas 3.3.2, 3.3.4 and the fact that $g(t) = 1$ on $[a_2,b_2]$, the implication (i) $\Rightarrow$ (ii) holds for $o < \tau \leq 1$.

Thus, following the proof of (i) $\Rightarrow$ (ii) of Theorem 2.4.1, assuming that the result holds for $\tau \in (p'-1,p')$ we have to prove it for $\tau \in [p',p'+1)$ $(p' = 1,2,\ldots, 2k+1)$.

Let $a_1^*, b_1^*, a_2^*, b_2^*$, $\chi_2(u)$ and g be as in the proof of (i) $\Rightarrow$ (ii) of Theorem 2.4.1. Then

$$\| S^{(p)}_{\lambda_n,k+1,m}(fg,t) - (fg)^{(p)}(t) \|_{C[a_2^*,b_2^*]}$$

$$\leq \| D^p[S_{\lambda_n,k+1,m}(g(t)(f(u)-f(t)),t)] \|_{C[a_2^*,b_2^*]}$$

(3.3.13)
$$+ \| D^p [S_{\lambda_n,k+1,m}(f(u)(g(u)-g(t)),t)] \|_{C[a_2^*,b_2^*]}$$

$$= I_1 + I_2, \text{ say.}$$

I_1 is easily estimated as follows : By Theorem 3.2.3 and the hypothesis that (i) holds we have

$$I_1 = ||D^p[\, S_{\lambda_n,k+1,m}(g(t)f(u),t)] - (fg)^{(p)}(t)\, ||_{C\,[a_2^*,b_2^*]}$$

$$= ||\sum_{i=o}^{p} \binom{p}{i} g^{(p-i)}(t) S^{(i)}_{\lambda_n,k+1,m}(f,t) - (fg)^{(p)}(t)||_{C\,[a_2^*,b_2^*]}$$

(3.3.14) (By Leibniz theorem)

$$= ||\sum_{i=o}^{p} \binom{p}{i} g^{(p-i)}(t) f^{(i)}(t) - (fg)^{(p)}(t)||_{C[\,a_2^*,b_2^*\,]} + O(\lambda_n^{-\tau/2})$$

$$= O(\lambda_n^{-\tau/2}) .$$

To estimate I_2, by Leibniz theorem and Theorem 3.2.2 we get

$$I_2 = || \; -\sum_{i=o}^{p-1} \binom{p}{i} g^{(p-i)}(t) \;\; S^{(i)}_{\lambda_n,k+1,m}(f,t)$$

(3.3.15)
$$+ \; S^{(p)}_{\lambda_n,k+1,m}(f(u)(g(u)-g(t))\, \chi_2(u),t)||_{C[a_2^*,b_2^*]} + o(\lambda_n^{-(k+1)})$$

$$= ||\, I_3+I_4|| _{C[a_2^*,b_2^*]} \; + o(\lambda_n^{-(k+1)}), \text{ say.}$$

Then, by Theorem 3.2.3

(3.3.16) $I_3 = -\sum_{i=o}^{p-1} \binom{p}{i} g^{(p-i)}(t) f^{(i)}(t) + O(\lambda_n^{-\tau/2}),$

uniformly in $t \in [\, a_2^*,b_2^*\,]$.

Since by the induction hypothesis $f^{(p'+p-1)}$ exists and $\in Lip(1-\delta\,; a_1^*,b_1^*)$ for any $\delta > o$, therefore we can write

$$I_4 = \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}).$$

$$.S_{\lambda_n}^{r+m-1}(f(u)(g(u)-g(t))\, \chi_2(u), u_{r+m-1})du_{r+m-1}$$

$$= \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p'+p-1} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}).$$

(3.3.17) $$. S_{\lambda_n}^{r+m-1}((u-t)^i(g(u)-g(t))\, \chi_2(u), u_{r+m-1})du_{r+m-1}$$

$$+ \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}).$$

$$. S_{\lambda_n}^{r+m-1}\left(\frac{f^{(p'+p-1)}(\xi) - f^{(p'+p-1)}(t)}{(p'+p-1)!}(u-t)^{p'+p-1}.\right.$$

$$\left..(g(u)-g(t))\, \chi_2(u), u_{r+m-1}\right)du_{r+m-1}$$

(ξ lying between u and t)

$$= I_5 + I_6, \text{ say.}$$

By Theorem 3.2.2, we have

$$I_5 = \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p'+p-1} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}).$$

(3.3.18) $$. S_{\lambda_n}^{r+m-1}((u-t)^i(g(u)-g(t)), u_{r+m-1})du_{r+m-1} + o(\lambda_n^{-(k+1)})$$

(uniformly in $t \in [a_2^*, b_2^*]$)

$$= I_7 + o(\lambda_n^{-(k+1)}), \text{ say.}$$

Since $g \in C_0^\infty$ therefore by the Taylor expansion of g we have

$$I_7 = \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p'+p-1} \frac{f^{(i)}(t)}{i!} \sum_{j=1}^{p'+p+1} \frac{g^{(j)}(t)}{j!} \cdot$$

$$\cdot \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}) S_{\lambda_n}^{r+m-1}((u-t)^{i+j}, u_{r+m-1}) du_{r+m-1}$$

(3.3.19)
$$+ \sum_{r=1}^{k+1} \frac{(-1)^{r+1}}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{i=o}^{p'+p-1} \frac{f^{(i)}(t)}{i!} \cdot$$

$$\cdot \int_A^B W^{(p)}(\lambda_n, t, u_{r+m-1}) S_{\lambda_n}^{r+m-1}(\epsilon(u,t)(u-t)^{i+p'+p+1},$$

$$u_{r+m-1}) du_{r+m-1} \text{ (where } \epsilon(u,t) \to o \text{ as } u \to t)$$

$$= I_8 + I_9, \text{ say.}$$

Then, by Lemma 1.3.3 and Theorem 3.2.2 we have

(3.3.20) $$I_8 = \sum_{j=1}^{p} \frac{g^{(j)}(t)}{j!} \frac{f^{(p-j)}(t)}{(p-j)!} p! + O(\lambda_n^{-(k+1)}),$$

uniformly in $t \in [a_2^*, b_2^*]$.

Also, by Schwarz inequality and Corollary 2.2.3 it is easily seen that

(3.3.21) $$I_9 = O(\lambda_n^{-(p'+1)/2}) = O(\lambda_n^{-\tau/2}),$$

uniformly in $t \in [a_2^*, b_2^*]$.

Finally, we estimate I_6. By mean value theorem, Lemma 1.5.6 and Hölder's inequality we have

$$||I_6||_{C[a_2^*,b_2^*]} \leq \sum_{r=1}^{k+1} \frac{1}{m\beta(m,r)} \binom{k+m+1}{k-r+1} \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} \lambda_n^{i+j} \Big|\Big| \frac{|q_{ij}^{[p]}(t)|}{(p(t))^p} \cdot$$

$$\cdot \int_A^B W(\lambda_n, t, u_{r+m-1}) |u_{r+m-1}-t|^j \cdot$$

$$\cdot S_{\lambda_n}^{r+m-1}\Big(\frac{|f^{(p'+p-1)}(\xi)-f^{(p'+p-1)}(t)|}{(p'+p-1)!} \cdot$$

$$\cdot |g'(\eta)||u-t|^{p'+p} \chi_2(u), u_{r+m-1}\Big) \cdot$$

$$\cdot du_{r+m-1}\Big|\Big|_{C[a_2^*,b_2^*]}$$

(3.3.22)
$$\leq M\, ||g'||_{C[a_2^*,b_2^*]} \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} \lambda_n^{i+j} \max_{1\leq r\leq k+1} || \cdot$$

$$\cdot \int_A^B W(\lambda_n, t, u_{r+m-1}) |u_{r+m-1}-t|^j \cdot$$

$$\cdot S_{\lambda_n}^{r+m-1}(|u-t|^{p'+p+1-\delta} \chi_2(u), u_{r+m-1}) \cdot$$

$$\cdot du_{r+m-1}||_{C[a_2^*,b_2^*]}$$

$$= O(\lambda_n^{-(p'+1-\delta)/2})$$

$$= O(\lambda_n^{-\tau/2}),$$

on having chosen $0 < \delta \leq p'+1-\tau (>0)$.

Combining the above estimates, we conclude that

$$|| S_{\lambda_n,k+1,m}^{(p)}(fg,t)-(fg)^{(p)}(t)||_{C[a_2^*,b_2^*]} = O(\lambda_n^{-\tau/2}).$$

Since supp $fg \subset (a_2^*, b_2^*)$, by Lemmas 3.3.2 and 3.3.4 we have $(fg)^{(p)} \in Liz(\alpha, k+1; a_2^*, b_2^*)$ which reduces to the required result i.e., $f^{(p)} \in Liz(\alpha, k+1; a_2, b_2)$ since $g(t) = 1$ on $[a_2, b_2]$. Hence (ii) holds.

This completes the proof of Theorem 3.3.1.

3.4 SATURATION THEOREM FOR $S^{(p)}_{\lambda,k+1,m}(\cdot, t)$

The saturation behaviour of the operators $S^{(p)}_{\lambda,k+1,m}$ is also analogous to that of the ordinary operators $S_{\lambda,k+1,m}$. For we have the following :

THEOREM 3.4.1 : If S_λ are regular, $k,m \in \mathbb{N}^o$; $p \in \mathbb{N}$ and $f \in D_\psi(A,B)$, then in the following statements, the implications (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii) and (iv)$\Rightarrow$ (v)$\Rightarrow$ (vi) are valid :

(i) $f^{(p)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1, b_1]} |S^{(p)}_{\lambda_n,k+1,m}(f,t) - f^{(p)}(t)| = O(1) ;$$

(ii) $f^{(2k+p+1)} \in A.C.[a_2, b_2]$ and $f^{(2k+p+2)} \in L_\infty[a_2, b_2]$;

(iii) $\lambda^{k+1} \| S^{(p)}_{\lambda,k+1,m}(f,t) - f^{(p)}(t) \|_{C[a_3,b_3]} = O(1)$;

$f^{(p)}$ exists on $[a_1, b_1]$ and

(iv) $$\lambda_n^{k+1} \sup_{t \in [a_1, b_1]} |S^{(p)}_{\lambda_n,k+1,m}(f,t) - f^{(p)}(t)| = o(1) ;$$

(v) $f \in C^{2k+p+2}[a_2, b_2]$ and $\sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t) f^{(j)}(t) = 0$, $t \in [a_2, b_2]$ where $Q(j,k+1,m,p,t)$ are the polynomials occuring in (3.2.2);

(vi) $\lambda^{k+1} \| S^{(p)}_{\lambda,k+1,m}(f,t) - f^{(p)}(t) \|_{C[a_3,b_3]} = o(1)$.

PROOF : If (i) holds, following the proof of Theorem 2.4.1, it is clear that $f^{(p)}$ is continuous on $[a_1,b_1]$. Also, (i)$\Longrightarrow$(ii) of Theorem (3.3.1) yields that $f^{(2k+p+1)}$ **exists and is** continuous on (a_1,b_1). Let a_1^*,a_2^*,b_1^*,b_2^* be as in the proof of (i)$\Longrightarrow$(ii) of Theorem 2.5.1. We can choose an f^* such that supp $f^*\subset(a_1,b_1)$, $f^*(t) = f(t)$ on $[a_1^*,b_1^*]$ and that f^* is 2k+p+1 times continuously differentiable on (a_1,b_1). By Theorem 3.2.2 and (i) we have

$$(3.4.1)\quad ||S^{(p)}_{\lambda_n,k+1,m}(f^*,t)-f^{*(p)}(t)||_{C\,[a_2^*,b_2^*]} = O(\lambda_n^{-(k+1)}).$$

Defining the spaces $C_0[a_1,b_1]$ and $C_0^\infty(a_2^*,b_2^*)$ as in the proof of (i)$\Longrightarrow$(ii) of Theorem 2.5.1, it follows that for a p times continuously differentiable function q_* with supp $q_*\subset[a_1,b_1]$ and for any $g\in C_0^\infty(a_2^*,b_2^*)$, by(2.5.3) we get

$$|\lambda^{k+1} < S^{(p)}_{\lambda,k+1,m}(q_*,t)-q_*^{(p)}(t),\ g(t) >|$$

$$(3.4.2)\qquad = |\lambda^{k+1} < S_{\lambda,k+1,m}(q_*,t)-q_*(t),g^{(p)}(t)>|$$

$$\leq M_1\ ||q_*||,\ \text{say},$$

for all λ sufficiently large, where the constant M_1 is independent of λ and q_*.

Since f^* is continuous on (a_1,b_1) and supp $f^*\subset(a_1,b_1)$, there exists a sequence $\{f_\sigma\}$ of 2k+p+2 times continuously differentiable functions on (A,B) with supp $f_\sigma\subset(a_1,b_1)$ and converging to f^* in the norm $||.||$ defined by (2.5.2).

Then, for any $g \in C_0^\infty(a_1, b_1)$ and each function f_σ, by Theorem 3.2.2. we have

$$\lim_{\lambda \to \infty} \lambda^{k+1} < S^{(p)}_{\lambda,k+1,m}(f_\sigma, t) - f_\sigma^{(p)}(t),\ g(t) >$$

$$= < \sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t) f_\sigma^{(j)}(t), g(t) >$$

$$= < f_\sigma(t), \sum_{j=1}^{2k+p+2} Q^*(j,k+1,m,p,t) g^{(j)}(t) >,$$

where $Q^*_{2k+p+2}(D) = \sum_{j=1}^{2k+p+2} Q^*(j,k+1,m,p,t)D^j$ denotes the operator adjoint to $Q_{2k+p+2}(D) = \sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t)D^j$.

Now, it is clear from (3.4.1) that there exists a sequence $\{n_q\}$ of natural numbers such that

$$(3.4.4) \quad \lim_{q \to \infty} < S^{(p)}_{\lambda_{n_q}, k+1,m}(f^*, t) - f^{*(p)}(t), g(t) >$$

$$= < h(t), g(t) > ,$$

for every $g \in C_0^\infty(a_2^*, b_2^*)$, where $h \in L_\infty[a_2^*, b_2^*]$ is a fixed function.

By (3.4.2), it follows that

$$\lim_{q \to \infty} \lambda_{n_q}^{k+1} | < S^{(p)}_{\lambda_{n_q},k+1,m}(f^* - f_\sigma, t), -(f^* - f_\sigma)^{(p)}(t), g(t) > |$$

$$(3.4.5) \qquad \leq M_1 || f^* - f_\sigma ||.$$

Hence, by (3.4.3-5)

$$< f^*(t), Q^*_{2k+p+2}(D) g > = \lim_{\sigma \to \infty} < f_\sigma(t), Q^*_{2k+p+2}(D) g >$$

$$= \lim_{\sigma \to \infty} [\lim_{q \to \infty} < S^{(p)}_{\lambda_{n_q}, k+1, m}(f^* - f_\sigma, t) - (f^* - f_\sigma)^{(p)}(t)$$

$$, g(t) > + < f_\sigma(t), Q^*_{2k+p+2}(D) g >]$$

$$= \lim_{q \to \infty} < S^{(p)}_{\lambda_{n_q}, k+1, m}(f^*, t) - f^{*(p)}(t) - f^{*(p)}(t), g(t) >$$

$$= < h(t), g(t) >.$$

Hence,

$$(3.4.6) \quad Q_{2k+p+2}(D) f^* = h(t),$$

as generalized functions.

Now, noticing the easily varifiable fact that

$$Q(2k+p+2, k+1, m, p, t) \equiv Q(2k+2, k+1, m, t)$$

and that the latter is positive for $t \in (A,B)$, as shown in the proof of Theorem 2.5.1 and regarding (3.4.6) as a first order differential equation for $f^{*(2k+p+1)}$, it follows that $f^{*(2k+p+1)} \in$ A.C.$[a_2^*, b_2^*]$ and that $f^{*(2k+p+2)} \in L_\infty[a_2^*, b_2^*]$. From this (ii) follows, since $[a_2, b_2] \subset [a_2^*, b_2^*]$ and f^* coincides with f on $[a_2, b_2]$. This completes the proof of the implication (i) $\Rightarrow$ (ii).

Also, (ii) $\Rightarrow$ (iii) follows by Theorem 3.2.3.

For (iv) $\Rightarrow$ (v), proceeding as in the proof of (i) $\Rightarrow$ (ii), we obtain

$$Q_{2k+p+2}(D) f^* = 0,$$

from which (v) is immediate.

Lastly (v) $\Rightarrow$ (vi) follows from Theorem 3.2.2.

This completes the proof of Theorem 3.4.1.

CHAPTER 4

ORDINARY AND SIMULTANEOUS APPROXIMATION WITH COMBINATIONS OF FIXED ITERATES

4.1 INTRODUCTION

In Chapters 2-3 we studied approximation properties of generalized Micchelli type combinations of iterates of exponential type operators of varying orders but with the same λ. In this chapter we consider combinations $S_{\lambda,p}(f,k,t)$ of iterates of exponential type operators of a fixed order but with varying λ defined by replacing $S_{d_j\lambda}$ $(j=o,1,..,k)$ in (1.2.8) by their p-th iterate $S^p_{d_j\lambda}$. Thus,

$$(4.1.1) \quad S_{\lambda,p}(f,k,t) = \sum_{j=o}^{k} C(j,k)\ S^p_{d_j\lambda}(f,t),$$

where $C(j,k)$ are as in (1.2.11). On taking $p = 1$, $S_{\lambda,p}(f,k,t)$ reduces to the combination $S_\lambda(f,k,t)$ studied by May.

In view of the detailed workings of the ordinary and simultaneous approximation results about the combinations $S_{\lambda,k,m}(f,t)$ in the previous chapters, in the present case of the operators $S_{\lambda,p}(f,k,t)$ we propose to obtain the results both in ordinary and simultaneous approximation in a common set up. Thus, in this chapter we study the convergence $S^{(m)}_{\lambda,p}(f,k,t) \to f^{(m)}(t)$, which for $m = o$ reduces to the ordinary approximation and for $m \in \mathbb{N}$ corresponds to the simultaneous approximation. For $m = o$ and $p = 1$ our results reduce to the results of May. Moreover, with $m \in \mathbb{N}$ and $p = 1$ the results

of this chapter give rise to Theorems 1.5.1-5 stated without proof in Chapter 1.

4.2 DIRECT THEOREMS

Throughout this chapter p remains an arbitrary but fixed natural number and ψ denotes a GTF for the p-th iterates $\{S_\lambda^p\}$ of exponential type operators. We recall that $\psi(u) = (1+u^2)^N$, $N > o$ is a GTF for $\{S_\lambda^p\}$. Other notational conventions remain as in the previous chapters.

Utilizing the fact that $\sum_{j=o}^{k} C(j,k) = 1$, we have the following basic convergence result :

<u>THEOREM</u> 4.2.1 : Let $f \in D_\psi(A,B)$. For $m = o$ if f is continuous at a point $t \in (A,B)$ and for $m \in \mathbb{N}$ if $f^{(m)}$ exists at a point $t \in (A,B)$, then

$$(4.2.1) \quad \lim_{\lambda \to \infty} S_{\lambda,p}^{(m)}(f,k,t) = f^{(m)}(t).$$

Also if $f^{(m)}$ exists and is continuous on $\langle a,b\rangle$ then (4.2.1) holds uniformly in $[a,b]$.

<u>PROOF</u> : For $m = o$, the result follows from Theorem 2.2.5 on taking $k = 1$ and $m = p-1$ there. For $m \in \mathbb{N}$ the result follows, in a similar manner, from Theorem 3.2.1.

. An asymptotic formula for $S_{\lambda,p}^{(m)}(f,k,t)$ is given in

<u>THEOREM</u> 4.2.2 : Let $f \in D_\psi(A,B)$ and $m \in \mathbb{N}^o$. If $f^{(2k+m+2)}$ exists at a point $t \in (A,B)$ then,

$$(4.2.2) \quad \lim_{\lambda \to \infty} \lambda^{k+1} [S_{\lambda,p}^{(m)}(f,k,t)-f^{(m)}(t)] = \sum_{j=m}^{2k+m+2} Q(j,k,m,p,t)f^{(j)}(t)$$

and

$$(4.2.3)\quad \lim_{\lambda\to\infty} \lambda^{k+1}\,[S^{(m)}_{\lambda,p}(f,k+1,m)-f^{(m)}(t)] = o,$$

where $Q(j,k,m,p,t)$ are certain polynomials in t.

Further, if $f^{(2k+m+2)}$ exists and is continuous on $\langle a,b\rangle$ then (4.2.2-3) hold uniformly in $[a,b]$.

PROOF : With the hypothesis, we can write

$$S^{(m)}_{\lambda,p}(f,k,t) = \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j\lambda,t,u_{p-1})\, S^{p-1}_{d_j\lambda}\Big(\sum_{i=o}^{2k+m+2} \frac{f^{(i)}(t)}{i!}(u-t)^i$$

$$+\, \epsilon(u,t)(u-t)^{2k+m+2},u_{p-1}\Big)du_{p-1}$$

(where $\epsilon(u,t)\to o$ as $u\to t$)

$$(4.2.4)\quad = \sum_{i=o}^{2k+m+2} \frac{f^{(i)}(t)}{i!} \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j\lambda,t,u_{p-1})S^{p-1}_{d_j\lambda}((u-t)^i,u_{p-1})du_{p-1}$$

$$+ \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j\lambda,t,u_{p-1})\,.$$

$$.\ S^{p-1}_{d_j\lambda}(\epsilon(u,t)(u-t)^{2k+m+2},u_{p-1})du_{p-1}$$

$$= I_1 + I_2, \text{ say.}$$

First we shall estimate I_2.

Defining $\chi_\delta(u)$ as in the proof of Theorem 3.2.1, we have

$$|I_2| \le \sum_{j=o}^{k} |C(j,k)| \int_A^B |W^{(m)}(d_j\lambda, t,u_{p-1})|\,.$$

$$.\ S^{p-1}_{d_j\lambda}(|\epsilon(u,t)||u-t|^{2k+m+2}\chi_\delta(u),u_{p-1})du_{p-1}$$

$$(4.2.5)\quad + \sum_{j=o}^{k} |C(j,k)| \int_A^B |W^{(m)}(d_j\lambda,t,u_{p-1})|\,.$$

$$.\ S^{p-1}_{d_j\lambda}(|\epsilon(u,t)||u-t|^{2k+m+2}(1-\chi_\delta(u)),u_{p-1})du_{p-1}$$

$$= I_3 + I_4, \text{ say.}$$

Then, proceeding as in the proof of Theorem 3.2.1 it can be easily shown that

(4.2.6) $$I_3 = \epsilon . O(\lambda^{-(k+1)}), \text{ and}$$

(4.2.7) $$I_4 = o(\lambda^{-(k+1)}).$$

Now, since $\epsilon > o$ is arbitrary. Hence,

(4.2.8) $$|I_2| = o(\lambda^{-(k+1)}).$$

To estimate I_1, we first consider the case when $m = o$. Then

$$I_1 = \sum_{i=o}^{2k+2} \frac{f^{(i)}(t)}{i!} \sum_{j=o}^{k} C(j,k)\, \mu^{[p]}_{d_j\lambda,i}(t).$$

Since $\mu^{[p]}_{d_j\lambda,i}(t)$ is a polynomial in t and $1/d_j\lambda$, by Corollary 2.2.3, we can write

$$\mu^{[p]}_{d_j\lambda,i}(t) = \frac{P_o(t)}{(d_j\lambda)^{[\frac{i+1}{2}]}} + \frac{P_1(t)}{(d_j\lambda)^{[\frac{i+1}{2}]+1}} + \cdots\cdots$$

for certain polynomials P_j's depending on p and i.

Then, clearly

$$\sum_{j=o}^{k} C(j,k)\, \mu^{[p]}_{d_j\lambda,i}(t)$$

$$= \frac{1}{\Delta} \begin{vmatrix} \frac{P_o(t)}{(d_o\lambda)^{[\frac{i+1}{2}]}} + \frac{P_1(t)}{(d_o\lambda)^{[\frac{i+1}{2}]+1}} + \cdots & d_o^{-1} & d_o^{-2} & \cdots & d_o^{-k} \\ \frac{P_o(t)}{(d_1\lambda)^{[\frac{i+1}{2}]}} + \frac{P_1(t)}{(d_1\lambda)^{[\frac{i+1}{2}]+1}} + \cdots & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{P_o(t)}{(d_k\lambda)^{[\frac{i+1}{2}]}} + \frac{P_1(t)}{(d_k\lambda)^{[\frac{i+1}{2}]+1}} \cdots & d_k^{-1} & d_k^{-2} & & d_k^{-k} \end{vmatrix}$$

$$= i!\, \lambda^{-(k+1)} \, [\, Q(i,k,o,p,t) + o(1)\,]\, , \text{ say.}$$

Thus,

$$(4.2.9)\quad I_1 = f(t) + \lambda^{-(k+1)} [\sum_{i=o}^{2k+2} Q(i,k,o,p,t)\, f^{(i)}(t) + o(1)]$$

and hence for $m = o$, (4.2.2) follows from (4.2.8–9).

To prove (4.2.2) for $m \in \mathbb{N}$, utilizing (4.2.9) and proceeding as in the proof of Theorem 3.2.2 it is easily seen that

$$(4.2.10)\quad I_1 = f^{(m)}(t) + \lambda^{-(k+1)} [\sum_{j=m}^{2k+m+2} Q(j,k,m,p,t) f^{(j)}(t) + o(1)]$$

for some polynomials $Q(j,k,m,p,t)$ in t.

Combining (4.2.8) and (4.2.10),(4.2.2) for $m \in \mathbb{N}$ follows. The assertion (4.2.3) can be proved along similar lines.

The uniformity assertion follows as in the proof of Theorem 2.3.1.

Also, we remark that there holds

$$Q(2k+m+2,k,m,p,t) = Q(2k+2,k,o,p,t),\ m \in \mathbb{N}$$

and that

$$Q(2k+2,k,o,p,t) = \frac{(-1)^k (2k+1)!!\ p^{k+1}(t)}{\prod_{i=o}^{k} d_i}\, .$$

These assertions can be readily verified from the above proof and Lemma 1.3.3.

The next result provides an estimate of the degree of approximation in $S^{(m)}_{\lambda,p}(f,k,t) \to f^{(m)}(t)$, $m \in \mathbb{N}^o$.

<u>THEOREM</u> 4.2.3 : Let $m \in \mathbb{N}^{o}$, $m \le q \le 2k+m+2$, $f \in D_{\psi}(A,B)$ and $f^{(q)}$ exist and be continuous on $\langle a,b\rangle$. Then,

$$(4.2.11)\quad \|S_{\lambda,p}^{(m)}(f,k,t)-f^{(m)}(t)\|_{C[a,b]} \le \max\{C\,\lambda^{-(q-m)/2}\cdot \omega(f^{(q)},\lambda^{-1/2}),\ C'\lambda^{-(k+1)}\},$$

where $C = C(k,m,p)$ and $C' = C'(k,m,p,f)$.

<u>PROOF</u> : If $u \in \langle a,b\rangle$ and $t \in [a,b]$, we have

$$f(u) = \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^{i} + \frac{f^{(q)}(\xi)-f^{(q)}(t)}{q!}(u-t)^{q},$$

where ξ lies between u and t. Hence, we can write

$$(4.2.12)\quad f(u) = \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^{i} + \frac{f^{(q)}(\xi)-f^{(q)}(t)}{q!}(u-t)^{q}\ \chi^{*}(u)$$
$$+ F(u,t)(1-\chi^{*}(u)),$$

where $\chi^{*}(u)$ denotes the characteristic function of $\langle a,b\rangle$ and

$$F(u,t) = f(u)-\sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^{i},$$

for all $u \in (A,B)$ and $t \in [a,b]$.

Now, operating by $S_{\lambda,p}^{(m)}(\cdot,k,t)$ on (4.2.12) we have

$$S_{\lambda,p}^{(m)}(f,k,t)$$
$$= \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!}\ S_{\lambda,p}^{(m)}((u-t)^{i},k,t)$$
$$(4.2.13)\qquad + S_{\lambda,p}^{(m)}\left(\frac{f^{(q)}(\xi)-f^{(q)}(t)}{q!}(u-t)^{q}\ \chi^{*}(u),k,t\right)$$
$$+ S_{\lambda,p}^{(m)}(F(u,t)(1-\chi^{*}(u)),k,t)$$
$$= I_{1} + I_{2} + I_{3},\ \text{say.}$$

Then, by Theorem 4.2.2

$$I_1 = \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=o}^{i} (-1)^{i-j} \binom{i}{j} t^{i-j} S_{\lambda,p}^{(m)} (u^j, k, t)$$

$$= \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=o}^{i} (-1)^{i-j} \binom{i}{j} t^{i-j} [\, D^m t^j + O(\lambda^{-(k+1)})\,]$$

(4.2.14)

$$= \sum_{i=o}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=o}^{i} (-1)^{i-j} \binom{i}{j} \binom{j}{m}\, m!\, t^{i-m}$$

$$+ O(\lambda^{-(k+1)})$$

$$= f^{(m)}(t) + O(\lambda^{-(k+1)})$$ (using the identity occuring in the proof of Theorem 3.2.2),

uniformly in $t \in [a,b]$.

Also, by Theorem 4.2.2 we have

(4.2.15) $$I_3 = o(\lambda^{-(k+1)}),$$

uniformly in $t \in [a,b]$.

Lastly, by Lemma 1.5.6, Schwarz inequality and Corollary 2.2.3

$$|I_2| \leq \sum_{j=o}^{k} |C(j,k)| \sum_{\substack{2r+s \leq m \\ r,s \geq o}} (d_j \lambda)^{r+s} \frac{|d_{rs}^{[m]}(t)|}{(p(t))^m} \int_A^B W(d_j\lambda, t, u_{p-1}) .$$

$$\cdot\ |u_{p-1} - t|^s\, S_{d_j\lambda}^{p-1}\Big(\frac{|f^{(q)}(\xi) - f^{(q)}(t)|}{q!} |u-t|^q \chi^*(u), u_{p-1}\Big).$$

$$\cdot\ du_{p-1}$$

$$\leq \frac{\omega(f^{(q)}, \lambda^{-1/2})}{q!} \sum_{j=0}^{k} |C(j,k)| \sum_{\substack{2r+s\leq m \\ r,s\geq 0}} (d_j\lambda)^{r+s} \frac{|q_{rs}^{[m]}(t)|}{(p(t))^m} \cdot$$

(4.2.16) $$\cdot \int_A^B W(d_j\lambda, t, u_{p-1}) |u_{p-1}-t|^s \, [S_{d_j\lambda}^{p-1}(|u-t|^q, u_{p-1})$$

$$+ \lambda^{-1/2} S_{d_j\lambda}^{p-1}(|u-t|^{q+1}, u_{p-1})] \, du_{p-1}$$

$$= \omega(f^{(q)}, \lambda^{-1/2}) . O(\lambda^{-(q-m)/2}),$$

uniformly in $t \in [a,b]$.

Combining these estimates of $I_1 - I_3$ we obtain the required result.

4.3 INVERSE THEOREM FOR $S_{\lambda,p}^{(m)}(\cdot,k,t)$

In this section we establish the following inverse theorem for the operators $S_{\lambda,p}^{(m)}(\cdot,k,t)$ $(m \in \mathbb{N}^0)$.

THEOREM 4.3.1 : Let $0 < \alpha < 2$ and $f \in D_\psi(A,B)$. Then, in the following statements the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) are true.

(i) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\sup_{t \in [a_1,b_1]} |S_{\lambda_n,p}^{(m)}(f,k,t) - f^{(m)}(t)| = O(\lambda_n^{-\alpha(k+1)/2}) ;$$

(ii) $f^{(m)} \in \mathrm{Liz}(\alpha, k+1; a_2, b_2)$;

(iii) (a) For $m' < \alpha(k+1) < m'+1$, $m' = 0,1,2,\ldots,2k+1$: $f^{(m'+m)}$ exists and $\in \mathrm{Lip}(\alpha(k+1)-m'; a_2, b_2)$,

(b) For $\alpha(k+1) = m'+1$, $m' = o,1,\ldots,2k$:
$f^{(m'+m)}$ exists and $\in \mathrm{Lip}^*(1;a_2,b_2)$;

(iv) $$||S^{(m)}_{\lambda,p}(f,k,t)-f^{(m)}(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}).$$

We first prove some auxiliary results.

In the following lemma $K_m(\xi, f)$ stands for the functional defined in (3.3.1), which for $m = o$ reduces to the functional $K(\xi,f;a',b')$ defined in Chapter 1.

LEMMA 4.3.2 : Let $a < a' < a'' < b'' < b' < b$. If $f^{(m)} \in C_o$ with supp $f \subset [a'',b'']$ and $||S^{(m)}_{\lambda_n,p}(f,k,t)-f^{(m)}(t)||_{C[a,b]} \leq M\,\lambda_n^{-\alpha(k+1)/2}$, then

(4.3.1) $$K_m(\xi,f) \leq M_o[\lambda^{-\alpha(k+1)/2} + \lambda^{k+1}\xi K_m(\lambda^{-(k+1)},f)].$$

Consequently, $K_m(\xi,f) \leq M'\xi^{\alpha/2}$ for some constant M' i.e., $f \in C^m_o(\alpha,k+1;a',b')$.

PROOF : As usual, it is enough to show that

(4.3.2) $$K_m(\xi,f) \leq M_o[\lambda_n^{-\alpha(k+1)/2} + \lambda_n^{k+1}\xi K_m(\lambda_n^{-(k+1)},f)],$$

for all n sufficiently large.

Since supp $f \subset [a'',b'']$, as in the proof of Lemma 2.4.2, by Theorem (3.2.2), there exists a function $h_n \in G^{(m)}$ such that for $i = m$ and $2k+m+2$

$$||h_n^{(i)}(t)-S^{(i)}_{\lambda_n,p}(f,k,t)||_{C[a,b]} \leq M_1\,\lambda_n^{-(k+1)},$$

for all n sufficiently large.

Therefore,

$$K_m(\xi , f) \leq 3M_1 \lambda_n^{-(k+1)} + ||f^{(m)}(t)-S^{(m)}_{\lambda_n,p}(f,k,t)||_{C[a',b']}$$

$$+ \xi [||S^{(m)}_{\lambda_n,p}(f,k,t)||_{C[a',b']} + ||S^{(2k+m+2)}_{\lambda_n,p}(f,k,t)||_{C[a',b']}]$$

Hence it is sufficient to show that there exists an M_2, such that, for each $g \in G^{(m)}$

$$||S^{(2k+m+2)}_{\lambda,p}(f,k,t)||_{C[a',b']}$$

(4.3.3) $$\leq M_2 \lambda^{k+1} [||f^{(m)}-g^{(m)}||_{C[a',b']} + \lambda^{-(k+1)}|| g^{(2k+m+2)}||_{C[a',b']}] .$$

In fact,

$$||S^{(2k+m+2)}_{\lambda,p}(f,k,t)||_{C[a',b']}$$

$$\leq \sum_{j=o}^{k} |C(j,k)| \; ||\int_A^B W^{(2k+m+2)}(d_j\lambda , t, u_{p-1}).$$

$$\cdot S^{p-1}_{d_j\lambda}((f-g)(u),u_{p-1})du_{p-1}||_{C[a',b']}$$

(4.3.4) $$+ \sum_{j=o}^{k} |C(j,k)| \; || \int_A^B W^{(2k+m+2)}(d_j\lambda , t, u_{p-1}).$$

$$\cdot S^{p-1}_{d_j\lambda} (g(u),u_{p-1})du_{p-1}||_{C[a',b']}$$

$$= I_1 + I_2, \text{ say.}$$

Then, by Lemma 1.5.6, Schwarz inequality and Corollary 2.2.3 we have

$$I_1 = \sum_{j=o}^{k} C(j,k) \,||\, \int_A^B \; W^{(2k+m+2)}(d_j\lambda \;, t, u_{p-1}) .$$

$$\cdot \; S_{d_j\lambda}^{p-1} \;\Big(\sum_{i=o}^{m-1} \frac{(f-g)^{(i)}(t)}{i!}(u-t)^i + \frac{(f-g)^{(m)}(\xi)}{m!}(u-t)^m, u_{p-1}\Big) .$$

$$\cdot \; du_{p-1} ||_{C[\,a',b'\,]} \quad (\,\xi \text{ lying between } u \text{ and } t)$$

$$(4.3.5) \qquad \leq \frac{||f^{(m)}-g^{(m)}||_{C[a',b']}}{m\,!} \sum_{j=o}^{k} |C(j,k)| \,.$$

$$|| \sum_{\substack{2r+s\leq 2k+m+2 \\ r,s\geq o}} (d_j\lambda\,)^{r+s} \; \frac{|q_{rs}^{[2k+m+2]}(t)|}{(p(t))^{2k+m+2}} \, .$$

$$\cdot \; \int_A^B W(d_j\lambda \;, t, u_{p-1}) |u_{p-1}-t|^s \;\; S_{d_j\lambda}^{p-1} \,(|u-t|^m, u_{p-1}) .$$

$$\cdot \; du_{p-1} ||_{C[a',b']} \quad (\text{as supp } f \cup \text{supp } g \subset [a',b']\;)$$

$$\leq \; M_3 \;\lambda^{k+1} ||f^{(m)}-g^{(m)}||_{C[a',b']} \;,$$

where M_3 is independent of g and f.

By the Taylor's expansion of g, we have

$$g(u) = \sum_{i=o}^{2k+m+1} \frac{g^{(i)}(t)}{i!}(u-t)^i + \frac{g^{(2k+m+2)}(\eta)}{(2k+m+2)!}(u-t)^{2k+m+2} ,$$

η lying between u and t .

Therefore, by Lemma 1.4.6, Schwarz inequality and Corollary 2.2.3

$$I_2 \leq \sum_{j=o}^{k} |C(j,k)| \, || \int_A^B |W^{(2k+m+2)}(d_j\lambda, t, u_{p-1})| .$$

$$\cdot \; S_{d_j\lambda}^{p-1} \left(\frac{|g^{(2k+m+2)}(\eta)|}{(2k+m+2)!} |u-t|^{2k+m+2}, u_{p-1}\right) du_{p-1} ||_{C[a',b']}$$

$$\text{(4.3.6)} \quad \leq \frac{||g^{(2k+m+2)}||_{C[a',b']}}{(2k+m+2)!} \, || \sum_{\substack{2r+s \leq 2k+m+2 \\ r,s \geq o}} (d_j\lambda)^{r+s} .$$

$$\cdot \frac{|q_{rs}^{[2k+m+2]}(t)|}{(p(t))^{2k+m+2}} \int_A^B W(d_j\lambda, t, u_{p-1}) |u_{p-1}-t|^s .$$

$$\cdot \; S_{d_j\lambda}^{p-1}(|u-t|^{2k+m+2}, u_{p-1}) du_{p-1} ||_{C[a',b']}$$

$$\leq M_4 \, || \, g^{(2k+m+2)} ||_{C[a',b']} .$$

Combining these estimates we obtain (4.3.2).

This completes the proof of the lemma.

LEMMA 4.3.3: Let $a<a'<a''<b''<b'<b$. If $f^{(m)} \in C_o$ with $\operatorname{supp} f \subset [a'',b'']$ and also $f \in C_o^m(\alpha,k+1;a',b')$, then

$$\text{(4.3.7)} \quad ||S_{\lambda,p}^{(m)}(f,k,t)-f^{(m)}(t)||_{C[a,b]} \leq M \lambda^{-\alpha(k+1)/2},$$

for all λ sufficiently large, where M does not depend on λ.

PROOF : For $g \in G^{(m)}$, we have

$$||S_{\lambda,p}^{(m)}(f,k,t)-f^{(m)}(t)||_{C[a,b]}$$

$$\text{(4.3.8)} \leq ||S_{\lambda,p}^{(m)}(f-g,k,t)||_{C[a,b]} + ||S_{\lambda,p}^{(m)}(g,k,t)-f^{(m)}(t)||_{C[a,b]}$$

$$= I_1 + I_2, \text{ say.}$$

Since supp $f \cup$ supp $g \subset [a',b']$, hence it is easily seen that

$$(4.3.9) \quad I_1 \leq M_1 \, ||f^{(m)}-g^{(m)}||_{C[a',b']} .$$

Also, by theorem 4.2.2, we have

$$I_2 \leq ||g^{(m)}-f^{(m)}||_{C[a',b']} + M_2\lambda^{-(k+1)} \sum_{j=m}^{2k+m+2} ||g^{(j)}||_{C[a',b']}$$

$$\leq ||g^{(m)}-f^{(m)}||_{C[a',b']} + M_3\lambda^{-(k+1)}(||g||_{C[a',b']}$$

$$+ ||g^{(2k+m+2)}||_{C[a',b']}),$$

where M_3 is a certain constant.

Hence,

$$||S^{(m)}_{\lambda,p}(f,k,t)-f^{(m)}(t)||_{C[a,b]} \leq M_4K_m(\lambda^{-(k+1)},f)$$

$$\leq M\lambda^{-\alpha(k+1)/2} .$$

This proves the lemma.

PROOF OF THEOREM 4.3.1 : As in the proofs of Theorems 2.4.1 and 3.3.1 it is enough to show the implications (ii)$\Longrightarrow$(iv) and (i)$\Longrightarrow$(ii).

First let us assume (ii). Let a',a'',b',b'' and g be as in the proof of the implication (ii)$\Longrightarrow$(iv) of Theorem 2.4.1. Then supp $fg \subset (a_2,b_2)$ and $(fg)^{(m)} \in Liz(\alpha,k+1;a_2,b_2)$ since $f^{(m)} \in Liz(\alpha,k+1;a_2,b_2)$. Hence by Lemmas 3.3.4 and 4.3.3 we have

$$||S^{(m)}_{\lambda,p}(fg,k,t)-(fg)^{(m)}(t)||_{C[a_2,b_2]} = O(\lambda^{-\alpha(k+1)/2}).$$

Consequently, since $g(t) = 1$ on $[a'',b'']$ and $[a_3,b_3] \subset (a'',b'')$,

$$(4.3.11)\quad ||S^{(m)}_{\lambda,p}(fg,k,t)-f^{(m)}(t)||_{C[a_3,b_3]} = O(\lambda^{-\alpha(k+1)/2}).$$

Now, we shall show that

$$(4.3.12)\quad ||S^{(m)}_{\lambda,p}(fg,k,t)-S^{(m)}_{\lambda,p}(f,k,t)||_{C[a_3,b_3]} = o(\lambda^{-(k+1)}).$$

Let δ and $\chi_1(u)$ be as in the proof of the implication (ii)$\Longrightarrow$(iv) of Theorem 3.3.1. Then, by Lemma 1.5.6, Schwarz inequality and Corollary 2.2.3 we have

$$||S^{(m)}_{\lambda,p}(fg-f,k,t)||_{C[a_3,b_3]}$$

$$= ||S^{(m)}_{\lambda,p}((fg-f)(u)\,\chi_1(u),k,t)||_{C[a_3,b_3]}$$

$$\leq \sum_{j=0}^{k} |C(j,k)|\; ||\sum_{\substack{2r+s\leq m\\ r,s\geq 0}} (d_j\lambda)^{r+s}\, \frac{|q^{[m]}_{rs}(t)|}{(p(t))^m}\,.$$

$$(4.3.13)\qquad \cdot \int_A^B W(d_j\lambda,t,u_{p-1})|u_{p-1}-t|^s\,.$$

$$\cdot\; S^{p-1}_{d_j\lambda}\left(\frac{M\,\Psi(u)(u-t)^{2m''}}{\delta^{2m''}},\,u_{p-1}\right)du_{p-1}||_{C[a_3,b_3]}$$

$$(m''\,(\in \mathbb{N}) > [\tfrac{2k+m+2}{2}] \text{ being arbitrary })$$

$$= O(\lambda^{-(m-2m'')/2}).$$

Thus, (4.3.12) follows.

Combining (4.3.11-12) we have (iv).

Next, let us assume (i). Proceeding as in the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 2.4.1 we have $f^{(m)} \in C[a_1, b_1]$. Putting $\tau = \alpha(k+1)$, we first consider the case $o < \tau \leq 1$. Let a', a'', b', b'' and g be as in the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 2.4.1.

For $t \in [a', b']$, with $D \equiv \frac{d}{dt}$, we have

$$S^{(m)}_{\lambda_n, p}(fg, k, t) - (fg)^{(m)}(t)$$

$$= D^m [S_{\lambda_n, p}((fg)(u) - (fg)(t), k, t)]$$

$$= D^m [S_{\lambda_n, p}(f(u)(g(u) - g(t)), k, t)] \tag{4.3.14}$$

$$+ D^m [S_{\lambda_n, p}(g(t)(f(u) - f(t)), k, t)]$$

$$= I_1 + I_2, \text{ say.}$$

To estimate I_1, by Leibniz theorem we have

$$I_1 = \sum_{j=o}^{k} C(j,k) \frac{\partial^m}{\partial t^m} \int_A^B W(d_j \lambda_n, t, u_{p-1}) .$$

$$\cdot \ S^{p-1}_{d_j \lambda_n} (f(u)(g(u) - g(t)), u_{p-1}) du_{p-1}$$

$$= \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{m} \binom{m}{i} \int_A^B W^{(i)}(d_j \lambda_n, t, u_{p-1}) .$$

$$\cdot \ \frac{\partial^{m-i}}{\partial t^{m-i}} [S^{p-1}_{d_j \lambda_n}(f(u)(g(u) - g(t)), u_{p-1})] du_{p-1}$$

Thus, (4.3.12) follows.

Combining (4.3.11-12) we have (iv).

Next, let us assume (i). Proceeding as in the proof of the implication (i)$\Longrightarrow$(ii) of Theorem 2.4.1 we have $f^{(m)} \in C[a_1, b_1]$. Putting $\tau = \alpha(k+1)$, we first consider the case $o < \tau \leq 1$. Let a', a'', b', b'' and g be as in the proof of the implication (i)$\Longrightarrow$(ii) of Theorem 2.4.1.

For $t \in [a', b']$, with $D \equiv \frac{d}{dt}$, we have

$$S^{(m)}_{\lambda_n, p}(fg,k,t) - (fg)^{(m)}(t)$$

$$= D^m [S_{\lambda_n, p}((fg)(u)-(fg)(t),k,t)]$$

$$= D^m [S_{\lambda_n, p}(f(u)(g(u)-g(t)),k,t)] \tag{4.3.14}$$

$$+ D^m [S_{\lambda_n, p}(g(t)(f(u)-f(t)),k,t)]$$

$$= I_1 + I_2, \text{ say.}$$

To estimate I_1, by Leibniz theorem we have

$$I_1 = \sum_{j=o}^{k} C(j,k) \frac{\partial^m}{\partial t^m} \int_A^B W(d_j \lambda_n, t, u_{p-1}).$$

$$\cdot \; S^{p-1}_{d_j \lambda_n}(f(u)(g(u)-g(t)), u_{p-1}) du_{p-1}$$

$$= \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{m} \binom{m}{i} \int_A^B W^{(i)}(d_j \lambda_n, t, u_{p-1}).$$

$$\cdot \; \frac{\partial^{m-i}}{\partial t^{m-i}} [S^{p-1}_{d_j \lambda_n}(f(u)(g(u)-g(t)), u_{p-1})] \, du_{p-1}$$

$$= - \sum_{i=o}^{m-1} \binom{m}{i} g^{(m-i)}(t)\, S^{(i)}_{\lambda_n, p}(f,k,t)$$

$$(4.3.15) \qquad + \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j\lambda_n, t, u_{p-1}) \cdot$$

$$\cdot g^{p-1}_{d_j\lambda_n} (f(u)(g(u)-g(t)),\, u_{p-1}) du_{p-1}$$

$$= I_3 + I_4, \text{ say.}$$

Then, by Theorem 4.2.3

$$(4.3.16) \quad I_3 = - \sum_{i=o}^{m-1} \binom{m}{i} g^{(m-i)}(t)\, f^{(i)}(t) + O(\lambda_n^{-\tau/2}),$$

uniformly in $t \in [a', b']$.

Next, we estimate I_4. By Taylor expansions of f and g we have

$$f(u) = \sum_{i=o}^{m} \frac{f^{(i)}(t)}{i!}(u-t)^i + o(u-t)^m$$

and

$$g(u) = \sum_{i=o}^{m+1} \frac{g^{(i)}(t)}{i!}(u-t)^i + o(u-t)^{m+1} .$$

Substituting these expressions in I_4 and using Theorem 4.2.2, Schwarz inequality and Corollary 2.2.3 we get

$$I_4 = \sum_{i=1}^{m} \frac{g^{(i)}(t)}{i!} \frac{f^{(m-i)}(t)}{(m-i)!} m! + O(\lambda_n^{-1/2})$$

$$(4.3.17) \qquad = \sum_{i=1}^{m} \binom{m}{i} g^{(i)}(t)\, f^{(m-i)}(t) + O(\lambda_n^{-\tau/2}),$$

uniformly in $t \in [a', b']$.

Lastly, by Leibniz theorem

$$\sum_{j=0}^{k} C(j,k) \sum_{i=\sigma}^{m} \binom{m}{i} \int_A^B W^{(i)}(d_j \lambda_n, t, u_{p-1}).$$

$$\cdot \frac{\partial^{m-i}}{\partial t^{m-i}} [S^{p-1}_{d_j \lambda_n}(g(t)(f(u)-f(t)), u_{p-1})] \, du_{p-1}$$

(4.3.18)
$$= \sum_{i=0}^{m} \binom{m}{i} g^{(m-i)}(t) \, S^{(i)}_{\lambda_n, p}(f,k,t) - (fg)^{(m)}(t)$$

$$= \sum_{i=0}^{m} \binom{m}{i} g^{(m-i)}(t) f^{(i)}(t) - (fg)^{(m)}(t) + O(\lambda_n^{-\tau/2})$$

(by Theorem 4.2.3 and the hypothesis that (i) holds)

$$= O(\lambda_n^{-\tau/2}) \text{ (uniformly in } t \in [a', b'] \text{).}$$

Combining these estimates we have

(4.3.19) $$||S^{(m)}_{\lambda_n, p}(fg,k,t) - (fg)^{(m)}(t)||_{C[a',b']} = O(\lambda_n^{-\tau/2}).$$

Thus, by Lemmas 4.2.1 and 3.3.4 we have

$(fg)^{(m)} \in \mathrm{Liz}(\alpha, k+1; a', b')$. Since $g(t) = 1$ on $[a_2, b_2]$ hence $f^{(m)} \in \mathrm{Liz}(\alpha, k+1; a_2, b_2)$ proving the implication (i)$\Longrightarrow$(ii) when $0 < \tau \le 1$.

Now, as in the proof of the implication (i)$\Longrightarrow$(ii) of Theorem 2.4.1 for a general τ, we assume that the result holds for $\tau \in (p'-1, p')$ and prove it for $\tau \in [p', p'+1)$ $(p' = 1,2,.., 2k+1)$. Since the result holds for $\tau \in (p'-1, p')$ therefore $f^{(p'+m-1)}$ exists and $\in \mathrm{Lip}(1-\delta; a_1^*, b_1^*)$ for any $\delta > 0$.

Let $a_1^*, b_1^*, a_2^*, b_2^*$, $\chi_2(u)$ and g be chosen as in the proof of the implication (i)$\Longrightarrow$(ii) of Theorem 2.4.1. Then,

$$\|S^{(m)}_{\lambda_n,p}(fg,k,t)-(fg)^{(m)}(t)\|_{C[a_2^*,b_2^*]}$$

$$\le \|D^m[S_{\lambda_n,p}(g(t)(f(u)-f(t)),k,t)]\|_{C[a_2^*,b_2^*]}$$

(4.3.20)
$$+\|D^m[S_{\lambda_n,p}(f(u)(g(u)-g(t)),k,t)]\|_{C[a_2^*,b_2^*]}$$

$$= I_1 + I_2, \text{ say.}$$

To estimate I_1, by Theorem 4.2.3 and the assumption that (i) holds, we have

$$I_1 = \|D^m[S_{\lambda_n,p}(g(t)(f(u),k,t)] - (fg)^{(m)}(t)\|_{C[a_2^*,b_2^*]}$$

$$= \|\sum_{i=0}^{m}\binom{m}{i}g^{(m-i)}(t)S^{(i)}_{\lambda_n,p}(f,k,t)-(fg)^{(m)}(t)\|_{C[a_2^*,b_2^*]}$$

(4.3.21)
$$= \|\sum_{i=0}^{m}\binom{m}{i}g^{(m-i)}(t)f^{(i)}(t)-(fg)^{(m)}(t)\|_{C[a_2^*,b_2^*]} + O(\lambda_n^{-\tau/2})$$

$$= O(\lambda_n^{-\tau/2}).$$

Also, by Leibniz theorem and Theorem 4.2.2 we have

$$I_2 = \| -\sum_{i=0}^{m-1}\binom{m}{i}g^{(m-i)}(t)\,S^{(i)}_{\lambda_n,p}(f,k,t)$$

$$+ S^{(m)}_{\lambda_n,p}(f(u(g(u)-g(t))\,\chi_2(u),k,t)\|_{C[a_2^*,b_2^*]}$$

(4.3.22)
$$+ o(\lambda_n^{-(k+1)})$$

$$= \|I_3+I_4\|_{C[a_2^*,b_2^*]} + o(\lambda_n^{-(k+1)}), \text{ say.}$$

By Theorem 4.2.3 we get

$$(4.3.23)\quad I_3 = -\sum_{i=o}^{m-1} \binom{m}{i} g^{(m-i)}(t) f^{(i)}(t) + O(\lambda_n^{-\tau/2}),$$

uniformly in $t \in [a_2^*, b_2^*]$.

By a Taylor expansion of f we have

$$I_4 = \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j \lambda_n, t, u_{p-1}) \cdot$$

$$\cdot\ S_{d_j \lambda_n}^{p-1} (f(u)(g(u)-g(t)) \chi_2(u), u_{p-1}) du_{p-1}$$

$$= \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{p'+m-1} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(m)}(d_j \lambda_n, t, u_{p-1}) \cdot$$

$$\cdot\ S_{d_j \lambda_n}^{p-1} ((u-t)^i (g(u)-g(t)) \chi_2(u), u_{p-1}) du_{p-1}$$

$$(4.3.24)\qquad + \sum_{j=o}^{k} C(j,k) \int_A^B W^{(m)}(d_j \lambda_n, t, u_{p-1}) \cdot$$

$$\cdot\ S_{d_j \lambda_n}^{p-1} \left(\frac{f^{(p'+m-1)}(\xi) - f^{(p'+m-1)}(t)}{(p'+m-1)!} (u-t)^{p'+m-1} \right.\cdot$$

$$\left. \cdot (g(u)-g(t)) \chi_2(u), u_{p-1} \right) du_{p-1}$$

(ξ lying between u and t)

$$= I_5 + I_6, \text{ say.}$$

By Theorem 4.2.2, we get

$$I_5 = \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{p'+m-1} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(m)}(d_j \lambda_n, t, u_{p-1}) \cdot$$

$$\cdot\ S^{p-1}_{d_j\lambda_n}\ ((u-t)^i(g(u)-g(t)),u_{p-1})du_{p-1}$$

(4.3.25) $$+\ o(\lambda_n^{-(k+1)})\ \text{(uniformly in } t\in[a_2^*,b_2^*]$$

$$=\ I_7 + o(\lambda_n^{-(k+1)}),\ \text{say.}$$

Since $g \in C_o^\infty$, therefore we can write

$$I_7 = \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{p'+m-1} \frac{f^{(i)}(t)}{i!} \sum_{r=1}^{p'+m+1} \frac{g^{(r)}(t)}{r!}\ .$$

$$\cdot\ \int_A^B W^{(m)}(d_j\lambda_n,t,u_{p-1})\ S^{p-1}_{d_j\lambda_n}((u-t)^{i+r},u_{p-1})du_{p-1}$$

(4.3.26) $$+\ \sum_{j=o}^{k} C(j,k) \sum_{i=o}^{p'+m-1} \frac{f^{(i)}(t)}{i!} \int_A^B W^{(m)}(d_j\lambda_n,t,u_{p-1}).$$

$$\cdot\ S^{p-1}_{d_j\lambda_n}\ (\epsilon(u,t)(u-t)^{i+p'+m+1},u_{p-1})du_{p-1}$$

$$\text{(where } \epsilon(u,t)\to o \ \text{ as } u\to t)$$

$$=\ I_8 + I_9,\ \text{say.}$$

Then, by Theorem 4.2.2 we get

$$I_8 = \sum_{r=1}^{m} \frac{g^{(r)}(t)}{r!} \frac{f^{(m-r)}(t)}{(m-r)!}\, m! + O(\lambda_n^{-(k+1)})$$

(4.3.27) $$= \sum_{r=1}^{m} \binom{m}{r} g^{(r)}(t)\, f^{(m-r)}(t) + O(\lambda_n^{-(k+1)}),$$

uniformly in $t \in [a_2^*,b_2^*]$.

Also, as in the proof of Theorem 3.2.1 it can be easily shown that

$$(4.3.28)\quad I_9 = O(\lambda_n^{-(p'+1)/2}) = O(\lambda_n^{-\tau/2}),$$

uniformly in $t \in [a_2^*, b_2^*]$.

Lastly,

$$||I_6||_{C[a_2^*,b_2^*]}$$

$$\leq \sum_{j=o}^{k} |C(j,k)| \sum_{\substack{2r+s\leq m \\ r,s\geq o}} (d_j\lambda_n)^{r+s} \; || \frac{|q_{rs}^{[m]}(t)|}{(p(t))^m} \cdot$$

$$\cdot \int_A^B W(d_j\lambda_n, t, u_{p-1}) |u_{p-1}-t|^s \cdot$$

$$\cdot \; S_{d_j\lambda_n}^{p-1} \left(\frac{|f^{(p'+m-1)}(\xi)-f^{(p'+m-1)}(t)|}{(p'+m-1)!} \cdot \right.$$

$$\cdot \; |g'(n)| \, |u-t|^{p'+m} \; \chi_2(u), u_{p-1}) du_{p-1} ||_{C[a_2^*,b_2^*]}$$

(by Lemma 1.5.6 and mean value theorem)

$$(4.3.29)\quad \leq M \, ||g'||_{C[a_2^*,b_2^*]} \max_{o\leq j\leq k} \Big[\sum_{\substack{2r+s\leq m \\ r,s\geq o}} (d_j\lambda_n)^{r+s} \cdot$$

$$\cdot \; || \frac{|q_{rs}^{m}(t)|}{(p(t))^m} \int_A^B W(d_j\lambda_n, t, u_{p-1}) |u_{p-1}-t|^s \cdot$$

$$\cdot \; S_{d_j\lambda_n}^{p-1}(|u-t|^{p'+m+1-\delta} \; \chi_2(u), u_{p-1}) \cdot$$

$$\cdot \; du_{p-1} ||_{C[a_2^*,b_2^*]}$$

$$= O(\lambda_n^{-(p'+1-\delta/2)}) \quad \text{(by Schwarz inequality and Corollary 2.2.3)}$$

$$= O(\lambda_n^{-\tau/2}),$$

by choosing δ such that $o < \delta \leq p'+1-\tau$.

Combining the above estimates we get

$$||S^{(m)}_{\lambda_n,p}(fg,k,t)-(fg)^{(m)}(t)||_{C[a_2^*,b_2^*]} = O(\lambda_n^{-\tau/2}).$$

Since supp $fg \subset (a_2^*,b_2^*)$, therefore by Lemmas 4.3.2 and 3.3.4, $(fg)^{(m)} \in Liz(\alpha,k+1;a_2^*,b_2^*)$. Since $g(t) = 1$ on $[a_2,b_2]$, it follows that $f^{(m)} \in Liz(\alpha,k+1;a_2,b_2)$.

This completes the proof of the Theorem 4.3.1.

4.4 <u>SATURATION THEOREM FOR</u> $S^{(m)}_{\lambda,p}(.,k,t)$

The saturation behaviour of the operators $S^{(m)}_{\lambda,p}(.,k,t)$ also is similar to that of the operators $S^{(p)}_{\lambda,k+1,m}(.,t)$.

<u>THEOREM</u> 4.4.1 : If S_λ are regular, $k,m, \in IN^o$ and $f \in D_\psi(A,B)$, then, in the following statements, the implications (i) $\Longrightarrow$(ii)$\Longrightarrow$(iii) and (iv) $\Longrightarrow$(v)$\Longrightarrow$(vi) are true :

(i) $f^{(m)}$ exists on $[a_1,b_1]$ and
$$\lambda_n^{k+1} \sup_{t\in[a_1,b_1]} |S^{(m)}_{\lambda_n,p}(f,k,t)-f^{(m)}(t)| = O(1) ;$$

(ii) $f^{(2k+m+1)} \in A.C.[a_2,b_2]$ and $f^{(2k+m+2)} \in L_\infty[a_2,b_2]$;

(iii) $$\lambda^{k+1} ||S^{(m)}_{\lambda,p}(f,k,t)-f^{(m)}(t)||_{C[a_3,b_3]} = O(1) ;$$

(iv) $f^{(m)}$ exists on $[a_1,b_1]$ and
$$\lambda_n^{k+1} \sup_{t\in[a_1,b_1]} |S^{(m)}_{\lambda_n,p}(f,k,t)-f^{(m)}(t)| = o(1) ;$$

(v) $f^{(2k+m+2)} \in C[a_2,b_2]$ and $\sum_{j=m}^{2k+m+2} Q(j,k,m,p,t)\, f^{(j)}(t)$
$= o$, $t \in [a_2,b_2]$ where $Q(j,k,m,p,t)$ are the polynomials occuring in (4.2.2);

(vi) $\lambda^{k+1} \| S^{(m)}_{\lambda,p}(f,k,t)-f^{(m)}(t)\|_{C[a_3,b_3]} = o(1).$

PROOF : First let us assume (i). Then, following the proof of Theorem 2.4.1, it is clear that $f^{(m)}$ is continuous on $[a_1,b_1]$. Also, in view of the implication (i) $\Rightarrow$ (iii) of Theorem 4.3.1 we have that $f^{(2k+m+1)}$ exists and is continuous on (a_1,b_1). Let $a_1^*, b_1^*, a_2^*, b_2^*$ be chosen as in the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 2.5.1. Then, we can choose a function f^* with supp $f^* \subset (a_1,b_1)$ such that $f^*(t) = f(t)$ on $[a_1,b_1]$ and that f^* is 2k+m+1 times continously differentiable on (a_1,b_1). By Theorem 4.2.2 it follows that

(4.4.1) $$\|S^{(m)}_{\lambda_n,p}(f^*,k,t)-f^{*(m)}(t)\|_{C[a_2^*,b_2^*]} = O(\lambda_n^{-(k+1)}).$$

Let the spaces $C_o[a_1,b_1]$ and $C_o^\infty(a_2^*,b_2^*)$ be as in the proof of (i) $\Longrightarrow$ (ii) of Theorem 2.5.1. Then, for each $q_* \in C_o^m[a_1,b_1]$ and $g \in C_o^\infty(a_2^*,b_2^*)$ by Theorem 1.4.7 we get

$$\lambda^{k+1} < S^{(m)}_{\lambda,p}(q_*,k,t) - q_*^{(m)}(t),\ g(t) >$$

$$= \lambda^{k+1} < S_{\lambda,p}(q_*,k,t) - q_*(t),\ g^{(m)}(t)>$$

$$= \lambda^{k+1} \sum_{j=o}^{k} C(j,k) < S^p_{d_j\lambda}(q_*,t) - q_*(t), g^{(m)}(t) >$$

$$= \lambda^{k+1} < q_*(u),\ \sum_{j=o}^{k} C(j,k) \sum_{i=1}^{p} \binom{p}{i} G^{[m]}_{i,k+1}(u,d_j\lambda)$$

$$+ o(\lambda^{-(k+1)}) >$$

(where $G^{[m]}_{i,k+1}$ are the analogues of $F_{i,k+1}$ in Theorem 1.4.7 for the function $g^{(m)}$ and the o-term holds uniformly on $[a_1,b_1]$)

$$= < q_*(u), \sum_{j=o}^{k} C(j,k) \sum_{i=1}^{p} \binom{p}{i} g^{[m]}_{i,k+1,k+1}(u)/ d_j^{k+1} + o(1) > .$$

Hence,

$$(4.4.2) \qquad | \lambda^{k+1} < S^{(m)}_{\lambda,p}(q_*,k,t) - q_*^{(m)}(t), g(t) > | \leq M \, ||q_*||,$$

for all λ sufficiently large, where the constant M is independent of λ and q_*.

Since f^* is continuous on (a_1,b_1), there exists a sequence $\{f_\sigma\}$ such that $f_\sigma \in C_o^{2k+m+2}(a_1,b_1)$ and that as $\sigma \to \infty$, $f_\sigma \to f^*$ in the norm $||.||$ defined by (2.5.2). Then, by Theorem 4.2.2, for any $g \in C_o^\infty(a_1,b_1)$ and each function f_σ we get

$$\lim_{\lambda \to \infty} \lambda^{k+1} < S^{(m)}_{\lambda,p}(f_\sigma,k,t) - f_\sigma^{(m)}(t), g(t) >$$

$$(4.4.3) \qquad = < \sum_{j=m}^{2k+m+2} Q(j,k,m,p,t) f_\sigma^{(j)}(t), g(t) >$$

$$= < f_\sigma(t), \sum_{j=1}^{2k+m+2} Q^*(j,k,m,p,t) g^{(j)}(t) > ,$$

where $Q^*_{2k+m+2}(D) = \sum_{j=1}^{2k+m+2} Q^*(j,k,m,p,t)D^j$ denotes the differential operator adjoint to $Q_{2k+m+2}(D) = \sum_{j=m}^{2k+m+2} Q(j,k,m,p,t)D^j$.

From (4.4.1) it is clear that there exists a subsequence $\{\lambda_{n_q}^{k+1} [S^{(m)}_{\lambda_{n_q},p}(f^*,k,t) - f^{*(m)}(t)] \}_{q=1}^{\infty}$ which converges to a function $h \in L_\infty[a_2^*,b_2^*]$ in the weak*-topology of the space $L_\infty[a_2^*,b_2^*]$. Thus for every $g \in C_o^\infty(a_2^*,b_2^*)$ we have

$$\lim_{q\to\infty} \lambda_{n_q}^{k+1} < S^{(m)}_{\lambda_{n_q},\,p}(f^*,k,t) - f^{*(m)}(t), g(t) >$$

$$= < h(t), g(t) > \qquad (4.4.4)$$

By (4.4.2), we have

$$\lim_{q\to\infty} \lambda_{n_q}^{k+1} \left| < S^{(m)}_{\lambda_{n_q},\,p}(f^* - f_\sigma, t) - (f^* - f_\sigma)^{(m)}(t), g(t) > \right|$$

$$\leq M \, \| f^* - f_\sigma \| . \qquad (4.4.5)$$

Thus, combining (4.4.3-5) we get

$$< f^*(t), Q^*_{2k+m+2}(D) g >$$

$$= \lim_{\sigma\to\infty} < f_\sigma(t), Q^*_{2k+m+2}(D) g >$$

$$= \lim_{\sigma\to\infty} \Big[\lim_{q\to\infty} \lambda_{n_q}^{k+1} < S^{(m)}_{\lambda_{n_q},\,p}(f^* - f_\sigma, t) - (f^* - f_\sigma)^{(m)}(t)$$

$$, g(t) > + < f_\sigma(t), Q^*_{2k+m+2}(D) g > \Big]$$

$$= \lim_{q\to\infty} \lambda_{n_q}^{k+1} < S^{(m)}_{\lambda_{n_q},\,p}(f^*, t) - f^{*(m)}(t), g(t) >$$

$$= < h(t), g(t) > .$$

Hence,

(4.4.6) $\quad Q_{2k+m+2}(D)\, f^* = h(t)$, as generalized functions.

Now, following the remark after Theorem 4.2.2 we have $Q(2k+m+2, k, m, p, t) \neq 0$. Therefore we can write (4.4.6) as a first order linear differential equation for $f^{*(2k+m+1)}$, from which we conclude that $f^{*(2k+m+1)} \in A.C.\,[a_2^*, b_2^*]$

and also that $f^{*(2k+m+2)} \in L_\infty[a_2^*, b_2^*]$. From this (ii) is immediate, since $[a_2, b_2] \subset [a_2^*, b_2^*]$ and f^* coincides with f on $[a_2, b_2]$.

Also, (ii) $\Rightarrow$(iii) is immediate from Theorem 4.2.3.

To prove (iv) $\Rightarrow$(v), assuming (iv) and proceeding as in the proof of (i) $\Rightarrow$(ii), we get

$$Q_{2k+m+2}(D)\, f^*(t) = o,$$

from which (v) is clear.

Finally, (v) $\Rightarrow$(vi) follows from Theorem 4.2.2.

This completes the proof of the saturation theorem.

CHAPTER 5

SATURATION THEOREMS FOR ITERATIVE COMBINATIONS OF BERNSTEIN POLYNOMIALS

5.1 INTRODUCTION

In the previous chapters we proved direct and inverse theorems for the two types of iterative combinations (both in ordinary and simultaneous approximation) of general exponential type operators and saturation theorems for regular exponential type operators. However, several important particular cases of exponential type operators do not satisfy the regularity condition. Especially this is so for the summation type operators S_λ^3 (Bernstein polynomials), S_λ^4 (Szász operators), S_λ^5 (Baskakov operators) and S_λ^6 (the operators of Rathore) as defined in Chapter 1.

So far it is not clear how the saturation theorems could be obtained for general non-regular exponential type operators ; although from the existence of the asymptotic formulae one is led to conjecture that similar saturation theorems hold even in the case of non-regular exponential type operators. May [27] verified this assertion in the ordinary approximation for the non-iterative combinations $S_\lambda(.,k,t)$ of the operators $S_\lambda^3 - S_\lambda^5$.

We, in this chapter, obtain the saturation theorems (both in ordinary and simultaneous approximation) for the iterative combinations $S_{\lambda,p}(\cdot,k,t)$ and $S_{\lambda,k,m}(\cdot,t)$ of the Bernstein polynomials. The corresponding saturation theorems for the Szász operators, the Baskakov operators and the operators of Rathore have also been obtained by us but these will be published elsewhere.

Throughout this chapter $S_\lambda(f,t) \equiv B_\lambda(f,t)$, $\lambda \in \mathbb{N}$, where $B_\lambda(f,t) \equiv S_\lambda^3(f,t)$, the λ-th Bernstein polynomial. As the domain of these operators is the set of all bounded complex valued functions on the interval $[0,1]$, the functions f occuring in this chapter are supposed to be bounded on $[0,1]$. Other notational conventions of the previous chapters remain valid as such. In particular, we mention that $\{\lambda_n; n \in \mathbb{N}\}$ is a monotonically divergent sequence of natural numbers with $\lambda_{n+1}/\lambda_n \leq C$, $n \in \mathbb{N}$, for some fixed constant C ; the iterative combinations $S_{\lambda,p}(\cdot,k,t)$ and $S_{\lambda,k+1,m}(\cdot,t)$ are defined as in (4.1.1) and (2.2.2), respectively and the numbers a_i and b_i $(i = 1,2,3)$ satisfy $0<a_1<a_2<a_3<b_3<b_2<b_1<1$.

5.2 SATURATION THEOREM FOR $S_{\lambda,p}(\cdot,k,t)$

The results of this chapter show that even though the Bernstein polynomials are not regular operators, their saturation behaviour is identical to that of the regular exponential type operators. The result for the combinations $S_{\lambda,p}(\cdot,k,t)$ is as follows :

THEOREM 5.2.1 : In the following statements the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) hold for any $p \in \mathbb{N}$ and $k \in \mathbb{N}^o$.

(i) $\lambda_n^{k+1} \sup_{t \in [a_1,b_1]} |S_{\lambda_n,p}(f,k,t)-f(t)| = O(1)$;

(ii) $f^{(2k+1)} \in A.C.\,[a_2,b_2]$ and $f^{(2k+2)} \in L_\infty[a_2,b_2]$;

(iii) $\lambda^{k+1} \| S_{\lambda,p}(f,k,t) - f(t) \|_{C[a_3,b_3]} = O(1)$;

(iv) $\lambda_n^{k+1} \sup_{t \in [a_1,b_1]} |S_{\lambda_n,p}(f,k,t)-f(t)| = o(1)$;

(v) $f \in C^{2k+2}[a_2,b_2]$ and $\sum_{j=o}^{2k+2} Q(j,k,p,t)f^{(j)}(t) = o, t \in [a_2,b_2]$,

where $Q(j,k,p,t)$ are the polynomials occuring in (4.2.2);

(vi) $\lambda^{k+1} \| S_{\lambda,p}(f,k,t) - f(t) \|_{C[a_3,b_3]} = o(1)$.

PROOF : We begin with the following :

PROPOSITION 5.2.2 : Given an arbitrary $\alpha > 1$, there exists a sequence $\{n_k \in \mathbb{N}\}_{k=1}^{\infty}$ such that

$$\alpha C \geq \frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} \geq \alpha, \; k \in \mathbb{N}.$$

Moreover, $\{\lambda_{n_k}\}_{k=1}^{\infty}$ can be assumed to be such that

$$\lim_{k \to \infty} \frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} = \beta,$$

for some $\beta \in [\alpha, \alpha C]$.

PROOF : We choose n_1 arbitrarily. Then, assuming that the string $\{n_i\}_{i=1}^{k}$ $(k = 1,2,\dots)$ has been chosen to satisfy the given requirement, it is sufficient to prove the existence of n_{k+1}. If, on the contrary, a choice of $\lambda_{n_{k+1}}$ is impossible, then for every $n > n_k$ there holds either $\lambda_n/\lambda_{n_k} < \alpha$ or $\lambda_n/\lambda_{n_k} > \alpha C$.

But $\{\lambda_n : n > n_k\}$ is an increasing sequence. Thus either $\lambda_{n_{k+1}}/\lambda_{n_k} > C\alpha$ (which is impossible as $\alpha > 1$) or there exists an $\bar{n} \in \mathbb{N}$ such that $\lambda_n/\lambda_{n_k} < \alpha$, for $n_k < n \leq \bar{n}$ and $\lambda_{\bar{n}+1}/\lambda_{n_k} > \alpha C$. Now we have $\lambda_{\bar{n}}/\lambda_{n_k} < \alpha$ and $\lambda_{\bar{n}+1}/\lambda_{n_k} > \alpha C$.

This, however, shows that $\lambda_{\bar{n}+1}/\lambda_{\bar{n}} > C$, which is a contradiction. The second assertion follows from the Bolzano-Weierstrass theorem. This completes the proof.

PROOF OF THE IMPLICATION (i) $\Rightarrow$ (ii) : If (i) holds, then as in the proof of (i) $\Rightarrow$ (ii) of(saturation) Theorem 4.4.1 it follows that f is continuous on $[a_1,b_1]$ and that $f^{(2k+1)}$ exists and is continuous on (a_1,b_1). Let a_1^*,a_2^*,b_1^*,b_2^* be such that $a_1 < a_1^* < a_2^* < a_2 < b_2 < b_2^* < b_1^* < b_1$. Then, there exists a function f^* such that supp $f^* \subset (a_1,b_1)$, $f^*(t) = f(t)$ on $[a_1^*,b_1^*]$ and that f^* is 2k+1 times continuously differentiable on $[0,1]$. Obviously we have

$$(5.2.1) \quad \|S_{\lambda_n,p}(f,k,t) - S_{\lambda_n,p}(f^*,k,t)\|_{C[a_2^*,b_2^*]} = O(\lambda_n^{-m}),$$

for every $m \in \mathbb{N}$.

Hence, as $f^*(t) = f(t)$ on $[a_1^*, b_1^*]$ and $[a_2^*, b_2^*] \subset (a_1^*, b_1^*)$

$$\begin{aligned}
&||S_{\lambda_n, p}(f^*, k, t) - f^*(t)||_{C[a_2^*, b_2^*]} \\
&\quad \leq ||S_{\lambda_n, p}(f^*, k, t) - S_{\lambda_n, p}(f, k, t)||_{C[a_2^*, b_2^*]} \\
&\qquad + ||S_{\lambda_n, p}(f, k, t) - f^*(t)||_{C[a_2^*, b_2^*]} \\
&\quad \leq O(\lambda_n^{-m}) + O(\lambda_n^{-(k+1)}).
\end{aligned}$$

Thus, taking $m (\in \mathbb{N}) \geq k+1$ we have

$$(5.2.2) \quad ||S_{\lambda_n, p}(f^*, k, t) - f^*(t)||_{C[a_2^*, b_2^*]} = O(\lambda_n^{-(k+1)}).$$

In particular, for $r \in \mathbb{N}$ (in the context of Proposition 5.2.2)

$$(5.2.3) \quad ||S_{\lambda_{n_r}, p}(f^*, k, t) - f^*(t)||_{C[a_2^*, b_2^*]} = O(\lambda_{n_r}^{-(k+1)}).$$

This statement is equivalent to

$$(5.2.4) \quad ||S_{\lambda_{n_{r+1}}, p}(f^*, k, t) - S_{\lambda_{n_r}, p}(f^*, k, t)||_{C[a_2^*, b_2^*]} = O(\lambda_{n_r}^{-(k+$$

For, trivially (5.2.3) $\Rightarrow$ (5.2.4).

Also, assuming (5.2.4), since $\lim_{r \to \infty} S_{\lambda_{n_r}, p}(f^*, k, t) = f^*(t)$, we can write

$$\begin{aligned}
f^*(t) = {} & S_{\lambda_{n_r}, p}(f^*, k, t) + [S_{\lambda_{n_{r+1}}, p}(f^*, k, t) - S_{\lambda_{n_r}, p}(f^*, k, t)] \\
& + [S_{\lambda_{n_{r+2}}, p}(f^*, k, t) - S_{\lambda_{n_{r+1}}, p}(f^*, k, t)] + \dots \\
& + [S_{\lambda_{n_{r+m}}, p}(f^*, k, t) - S_{\lambda_{n_{r+m-1}}}, p(f^*, k, t)] + \dots
\end{aligned}$$

Hence,

$$\| f^*(t) - S_{\lambda_{n_r}, p}(f^*, k, t) \|_{C[a_2^*, b_2^*]}$$

$$\leq \| S_{\lambda_{n_{r+1}}, p}(f^*, k, t) - S_{\lambda_{n_r}, p}(f^*, k, t) \|_{C[a_2^*, b_2^*]}$$

$$+ \| S_{\lambda_{n_{r+2}}, p}(f^*, k, t) - S_{\lambda_{n_{r+1}}, p}(f^*, k, t) \|_{C[a_2^*, b_2^*]}$$

$$+ \cdots + \| S_{\lambda_{n_{r+m}}, p}(f^*, k, t) - S_{\lambda_{n_{r+m-1}}, p}(f^*, k, t) \|_{C[a_2^*, b_2^*]}$$

$$+ \cdots\cdots$$

$$\leq M_1 \left(1/\lambda_{n_r}^{k+1} + 1/\lambda_{n_{r+1}}^{k+1} + \cdots + 1/\lambda_{n_{r+m-1}}^{k+1} + \cdots \right)$$

$$\leq \frac{M_1}{\lambda_{n_r}^{k+1}} \left(1 + \frac{1}{\alpha^{k+1}} + \frac{1}{\alpha^{2(k+1)}} + \cdots + \frac{1}{\alpha^{(m-1)(k+1)}} + \cdots \right)$$

$$= \frac{M_1}{\lambda_{n_r}^{k+1}} \; \frac{1}{1-(1/\alpha^{k+1})} = \frac{M_2}{\lambda_{n_r}^{k+1}},$$

where $M_2 = \dfrac{M_1}{1-(1/\alpha)^{k+1}}$, showing that (5.2.3) holds.

Thus, we may assume that

$\{ \lambda_{n_r}^{k+1} (S_{\lambda_{n_{r+1}}, p}(f^*, k, t) - S_{\lambda_{n_r}, p}(f^*, k, t)) \}$ is bounded

as a sequence in $C[a_2^*, b_2^*]$ and hence in $L_\infty[a_2^*, b_2^*]$. Then there exists an $h \in L_\infty[a_2^*, b_2^*]$ and a sequence $\{r_q\}_{q=1}^{\infty}$ of natural numbers such that

$\{ \lambda_{n_{r_q}}^{k+1} (S_{\lambda_{n_{r_q+1}}}, p(f^*, k, t) - S_{\lambda_{n_{r_q}}}, p(f^*, k, t)) \}$ converges to

h in the weak* topology of $L_\infty[a_2^*,b_2^*]$. In particular, for any $g \in C_o^\infty$ with supp $g \subset (a_2^*,b_2^*)$ we have

$$(5.2.5)\quad \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}},p}(f^*,k,t)\cdot S_{\lambda_{n_{r_q}},p}(f^*,k,t),g(t) >$$

$$\rightarrow < h(t),g(t) >.$$

Another intermediate result is in order.

LEMMA 5.2.3 : Let $g \in C_o^\infty(a_2^*,b_2^*)$ and $f \in C_o^{2k+1}(a_1,b_1)$. Then there holds the inequality

$$(5.2.6)\quad \begin{aligned} &| \lambda_{n_r}^{k+1} < S_{\lambda_{n_{r+1}},p}(f,k,t) - S_{\lambda_{n_r},p}(f,k,t),g(t) > | \\ &\qquad \leq M \; ||f||_{C^{2k+1}}, \end{aligned}$$

for all r sufficiently large, where M does not depend on f and r and

$$||f||_{C^{2k+1}} = ||f|| + ||f^{(2k+1)}||.$$

PROOF : Let $W^{[p]}(\lambda,t,u)$ denote the distributional kernel of the p-th iterate S_λ^p. We have

$$\begin{aligned} &< S_{\lambda_{n_r},p}(f,k,t),g(t) > \\ &= < \sum_{j=o}^{k} C(j,k)\; S^p_{d_j\lambda_{n_r}}(f,t),g(t) > \\ &= \int_o^1 \int_o^1 \{ \sum_{j=o}^{k} C(j,k)\; W^{[p]}(d_j\lambda_{n_r},t,u)f(u)g(t) \}du\,dt \\ &= \int_o^1 \int_{\text{supp } f} \{ \ldots. \}\; du\,dt \\ &= \int_o^1 \int_{\text{supp } f} \sum_{i=o}^{2k+2} \sum_{j=o}^{k} \frac{1}{i!} C(j,k)\; W^{[p]}(d_j\lambda_{n_r},t,u)f(u)g^{(i)}(u) \\ &\qquad .(t-u)^i\; du\,dt \end{aligned}$$

$$+ \int_o^1 \int_{\text{supp } f} \sum_{j=o}^{k} C(j,k)\, W^{[p]}(d_j\lambda_{n_r}, t, u)\, f(u)\, \varepsilon(t,u)(t-u)^{2k+2} du\, d$$

$= I_1 + I_2$, say, where $\varepsilon(t,u)\,(t-u)^{2k+2}$ is the remainder term corresponding to the partial Taylor expansion of g.

We can simplify I_1 as follows :

$$I_1 = \int_{\text{supp } f} \int_o^1 \sum_{i=o}^{2k+2} \sum_{j=o}^{k} \frac{1}{i!}\, C(j,k)\, W^{[p]}(d_j\lambda_{n_r}, t, u) f(u) g^{(i)}(u).$$

$$.(t-u)^i\, dt\, du$$

$$= \sum_{i=o}^{2k+2} \int_{\text{supp } f} \int_o^1 \sum_{j=o}^{k} C(j,k)\, W^{[p]}(d_j\lambda_{n_r}, t, u)\, \phi_i(u) t^i\, dt\, du,$$

where $\phi_i(u) = \sum_{m=i}^{2k+2} (-1)^{m-i} \frac{1}{i!(m-i)!} f(u) g^{(i)}(u)\, u^{m-i}$.

Before proceeding further we first prove an auxiliary lemma. A full force of the lemma will not be used in the present section. However, the intermediate steps in its proof will be of use.

LEMMA 5.2.4 : If $h \in C_o^{2k+1}$, supp $h \subset (a,b) \subset (o,1)$, then for every $p \in \mathbb{N}$ and $m \in \mathbb{N}^o$ there holds

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda, t, u) t^m\, dt\, du$$

$$= \int_o^1 h(t) \{ \sum_{j=o}^{2k+1} \frac{(-1)^j}{j!} [t^m \mu^{[p]}_{\lambda,j}(t)]^{(j)} \}\, dt + O(\lambda^{-(k+1)}) ||h||_{C^{2k+1}} \tag{5.2.7}$$

where the $O(\lambda^{-(k+1)})$-term does not depend on h.

PROOF : First we shall show that with p and m as in the statement of the lemma it is possible to have the following type of expansion

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda, t, u)\, t^m\, dt\, du$$

$$= C_o(p,m,h) + \frac{C_1(p,m,h)}{\lambda} + \dots + \frac{C_k(p,m,h)}{\lambda^k} + O(\lambda^{-(k+1)}) ||h||_{C^{2k}} \tag{5.2.8}$$

the $O(\lambda^{-(k+1)})$-term being independent of h.

We shall prove (5.2.8) by induction. Therefore first we consider the case when m = o and $p \in IN$. We prove that

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda,t,u)\, dt\, du$$

(5.2.9)
$$= \left(\frac{\lambda}{\lambda+1}\right)^p \int_a^b h(u)\, du + O(\lambda^{-(k+1)}) \, ||h||_{C^{2k+1}},$$

which obviously reduces to the required form

$$C_o(p,o,h) + \frac{C_1(p,o,h)}{\lambda} + \ldots\ldots + \frac{C_k(p,o,h)}{\lambda^k} + O(\lambda^{-(k+1)}) \, ||h||_{C^{2k+1}}$$

We have

$$W(\lambda,t,u) = \sum_{k_o=o}^{\lambda} \binom{\lambda}{k_o} t^{k_o}(1-t)^{\lambda-k_o}\, \delta\left(u-\frac{k_o}{\lambda}\right)$$

hence,

$$W^{[2]}(\lambda,t,u) = \sum_{k_1=o}^{\lambda} \binom{\lambda}{k_1} t^{k_1}(1-t)^{\lambda-k_1} \sum_{k_o=o}^{\lambda} \binom{\lambda}{k_o} \left(\frac{k_1}{\lambda}\right)^{k_o} \cdot$$
$$\cdot \left(1-\frac{k_1}{\lambda}\right)^{\lambda-k_o} \delta\left(u-\frac{k_o}{\lambda}\right),$$

and in general

$$W^{[p]}(\lambda,t,u) = \sum_{k_{p-1}=o}^{\lambda} \binom{\lambda}{k_{p-1}} t^{k_{p-1}}(1-t)^{\lambda-k_{p-1}} \sum_{k_{p-2}=o}^{\lambda} \sum_{k_{p-3}=o}^{\lambda} \ldots$$
$$\ldots \sum_{k_o=o}^{\lambda} \prod_{i=o}^{p-2} \left\{ \binom{\lambda}{k_i} \left(\frac{k_{i+1}}{\lambda}\right)^{k_i} \left(1-\frac{k_{i+1}}{\lambda}\right)^{\lambda-k_i} \right\} \delta\left(u-\frac{k_o}{\lambda}\right).$$

Hence,

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda,t,u)\, dt\, du$$

$$= \sum_{k_{p-1}=o}^{\lambda} \frac{1}{\lambda+1} \sum_{k_{p-2}=o}^{\lambda} \ldots \sum_{k_o=o}^{\lambda} \prod_{i=o}^{p-2} \left\{ \binom{\lambda}{k_i} \left(\frac{k_{i+1}}{\lambda}\right)^{k_i} \left(1-\frac{k_{i+1}}{\lambda}\right)^{\lambda-k_i} \right\} h\left(\frac{k_o}{\lambda}\right.$$

$$= \sum_{k_{p-1}=0}^{\lambda} \frac{1}{\lambda+1} S_\lambda^{[p-1]}\left(h,\frac{k_{p-1}}{\lambda}\right) \text{ (Here after } S_\lambda^{j} \equiv S_\lambda^{[j]} \text{)}$$

(5.2.10)

$$= \frac{\lambda}{\lambda+1} \left(\int_0^1 S_\lambda^{[p-1]}(h,t)dt + R\right), \text{ say,}$$

where for k = o, an elementry computation shows that

$$R = \frac{1}{2\lambda} \left[S_\lambda^{[p-1]}(h,o) + S_\lambda^{[p-1]}(h,1) + S \right] ,$$

where $|S| \leq || S_\lambda^{[p-1](1)} h||$; and for $k \neq o$, by the classical Euler-Maclaurin sum formula

$$\begin{aligned} R = {} & \frac{1}{2\lambda}\left(S_\lambda^{[p-1]}(h,o) + S_\lambda^{[p-1]}(h,1)\right) \\ & + \frac{B_2}{2!\,\lambda^2}\left(S_\lambda^{[p-1](1)}(h,1) - S_\lambda^{[p-1](1)}(h,o)\right) \\ & + \frac{B_4}{4!\,\lambda^4}\left(S_\lambda^{[p-1](3)}(h,1) - S_\lambda^{[p-1](3)}(h,o)\right) \\ & + \cdots + \frac{B_{2k-2}}{(2k-2)!\,\lambda^{2k-2}} \left(S_\lambda^{[p-1](2k-3)}(h,1) - S_\lambda^{[p-1](2k-3)}(h,o)\right) \\ & - \frac{1}{(2k)!\,\lambda^{2k+1}} \sum_{m=o}^{\lambda-1} \int_0^1 \beta_{2k}(t)\, S_\lambda^{[p-1](2k)}(h, \lambda^{-1}(t+m))\, dt, \end{aligned}$$

where $\beta_{2k}(t) = B_{2k}(t) - B_{2k}$, $B_{2k}(t)$ and B_{2k} denoting respectively the 2k-th Bernoulli polynomial and the 2k-th Bernoulli number.

Now we shall show that

(5.2.11) $$S_\lambda^{[p-1](r)}(h,t) = O(\lambda^{-(k+1)})\,||h||_{C^{2k+1}},$$

for t = o,1 and for every $r (\in \mathbb{N})$ such that $o \leq r \leq 2k+1$.

We shall show this for t = o. For t = 1 it then follows by the symmetry in the definition of the Bernstein polynomials.

Since supp $h \subset (a,b)$, we have

$$(5.2.12) \quad h^{(r)}(u) \leq ||h^{(r)}|| \left(\frac{u}{a}\right)^m$$

$$(5.2.13) \quad -h^{(r)}(u) \leq ||h^{(r)}|| \left(\frac{u}{a}\right)^m,$$

where $m \ (\in \mathbb{N}) \geq 2k+2$ is arbitrary.

We define

$$g(u) = \frac{||h^{(r)}|| \, u^{m+r}}{a^m(m+r)\,(m+r-1)\ldots.(m+1)}.$$

Then (5.2.12) and (5.2.13) yield

$$g^{(r)}(u) - h^{(r)}(u) \geq o \text{ and } g^{(r)}(u) + h^{(r)}(u) \geq o .$$

From these (e.g., by [23,p.12]) we easily conclude that

$$S_\lambda^{[p-1](r)}(g-h,t) \geq o \quad \text{and} \quad S_\lambda^{[p-1](r)}(g+h,t) \geq o.$$

Consequently,

$$(5.2.14) \quad |S_\lambda^{[p-1](r)}(h,t)| \leq S_\lambda^{[p-1](r)}(g,t).$$

Now,

$$(5.2.15) \quad S_\lambda^{[p-1]}(u^{m+r},t) = t^{m+r} + \sum_{j=1}^{m+r} \binom{m+r}{j} t^{m+r-j} \mu_{\lambda,j}^{[p-1]}(t).$$

Hence, by (2.2.8) we get

$$S_\lambda^{[p-1](r)}(u^{m+r},o) = O(\lambda^{-(k+1)}).$$

Therefore, by (5.2.14)

$$|S_\lambda^{[p-1](r)}(h,o)| \leq ||h^{(r)}|| \, O(\lambda^{-(k+1)})$$

$$(5.2.16) \qquad \leq ||h||_{C^{2k+1}} \, O(\lambda^{-(k+1)}).$$

Also, from (5.2.14) and (5.2.15) it is clear that

$$(5.2.17)\qquad ||S_\lambda^{[p-1](1)}h|| \leq ||h||_{C^{2k+1}}\, O(1),$$

which together with (5.2.16), for k = o, implies that

$$|R| = O(1/\lambda)\ ||h||_{C'}.$$

For $k \neq o$, using (5.2.14-16) we have

$$|R| = O(1/\lambda^{k+1})\ ||h||_{C^{2k+1}}.$$

Thus for all $k \in \mathbb{N}^o$

$$(5.2.18)\qquad |R| = O(1/\lambda^{k+1})\ ||h||_{C^{2k+1}}.$$

Now suppose that the result (5.2.9) is true for p-1. Then, by (5.2.10) and (5.2.18)

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda,t,u)\, dt\, du = \left(\frac{\lambda}{\lambda+1}\right)^p \int_a^b h(u)du + O(\lambda^{-(k+1)})||h||_{C^{2}}$$

showing that (5.2.9) is true for p. Thus to prove (5.2.9) for all $p \in \mathbb{N}$ it is sufficient to establish it for p = 1.

For this, since supp $h \subset (a,b)$,

$$\int_a^b h(u) \int_o^1 W(\lambda,t,u)\, dt\, du = \sum_{m=o}^{\lambda} \frac{1}{\lambda+1}\, h\left(\frac{m}{\lambda}\right)$$

$$= \frac{\lambda}{\lambda+1} \cdot \left[\int_a^b h(u)du + R \right],$$

where, as previously, for k = o,

$$|R| \leq \frac{1}{2\lambda}\, ||h'||,$$

and for $k \neq o$,

$$R = - \frac{1}{2k!\ \lambda^{2k+1}} \sum_{m=o}^{\lambda-1} \int_o^1 \beta_{2k}(t)\ h^{(2k)}(\lambda^{-1}(t+m))dt,$$

$$= O(\lambda^{-(k+1)})\ ||h||_{C^{2k+1}}.$$

Hence for all $k \in \mathbb{N}^o$,

$$|R| = O(\lambda^{-(k+1)})\ ||h||_{C^{2k+1}}.$$

Thus, (5.2.9) has been proved for all $p \in \mathbb{N}$, i.e. (5.2.8) holds for $m = o$ and all $p \in \mathbb{N}$.

Now we consider the case when $p = 1$ and $m \in \mathbb{N}^o$. We prove that

$$\int_a^b h(u) \int_o^1 W(\lambda,t,u)\ t^m\ dt\ du$$

(5.2.19)

$$= \left(\frac{\lambda}{\lambda+1}\right) \int_a^b h(u)\ P_m(u,\lambda)\ du + O(\lambda^{-(k+1)})\ ||h||_{C^{2k+1}},$$

where $P_m(u,\lambda)$ is a polynomial in u and λ^{-1}, the degree of P_m in u being $\leq m$ and which reduces to the required form

$$C_o(1,m,h) + \frac{C_1(1,m,h)}{\lambda} + \ldots + \frac{C_k(1,m,h)}{\lambda^k} + O(\lambda^{-(k+1)})\ ||h||_{C^{2k+1}}.$$

Suppose that (5.2.19) is true for some m. Then,

$$\int_a^b h(u) \int_o^1 W(\lambda,t,u)\ t^{m+1}\ dt\ du$$

$$= \int_a^b h(u)u \int_o^1 W(\lambda,t,u)\ t^m\ dt\ du$$

$$- \int_a^b h(u) \int_o^1 W(\lambda,t,u)\ (u-t)\ t^m\ dt\ du$$

(5.2.20)

$$= \int_a^b h(u)u \int_o^1 W(\lambda,t,u)t^m \, dt\, du$$

$$- \frac{1}{\lambda} \int_a^b h(u) \int_o^1 -[\frac{\partial}{\partial t} W(\lambda,t,u)] \, t^{m+1}(1-t) \, dt\, du$$

$$= \int_a^b h(u)u \int_o^1 W(\lambda,t,u) \, t^m \, dt\, du$$

$$+ \frac{1}{\lambda} \int_a^b h(u) \int_o^1 W(\lambda,t,u) \, t^m [\,(m+1)-(m+2)t\,] \, dt\, du .$$

Now, (5.2.20) gives rise to

$$\int_a^b h(u) \int_o^1 W(\lambda,t,u) \, t^{m+1} \, dt\, du$$

$$= (\frac{\lambda}{\lambda+1}) \int_a^b h(u) \, P_{m+1}(u,\lambda) \, du + O(\lambda^{-(k+1)}) \, ||h||_{C^{2k+1}}$$

where

$$P_{m+1}(u,\lambda) = [1 - \frac{m+2}{\lambda} + \ldots\ldots + (-1)^k (\frac{m+2}{\lambda})^k][\,u P_m(u,\lambda) + \frac{m+1}{\lambda} P_m(u,\lambda)]$$

is of the required type.

Thus, since we have already proved it for $m = o$, (5.2.9) holds for all $m \in \mathbb{N}^o$. Consequently (5.2.8) holds for $p = 1$ and all $m \in \mathbb{N}^o$ and also for $m = o$ and all $p \in \mathbb{N}$.

Lastly, we prove (5.2.8) for all $p \in \mathbb{N}$ and $m \in \mathbb{N}^o$. For this suppose that (5.2.8) holds for $p-1$ and upto some $m \in \mathbb{N}$. Then,

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda,t,u) \, t^m \, dt\, du$$

$$= \sum_{k_{p-1}=o}^{\lambda} \frac{(k_{p-1}+1)(k_{p-1}+2)\ldots(k_{p-1}+m)}{(\lambda+1)\,(\lambda+2)\ldots(\lambda+m+1)} \; S_\lambda^{[p-1]}\,(h,\frac{k_{p-1}}{\lambda})$$

(5.2.21)

$$= \frac{1}{\lambda+1} \cdot \frac{\lambda^m}{(\lambda+2)(\lambda+3)\ldots(\lambda+m+1)} \sum_{k_{p-1}=o}^{\lambda} Q_m(\frac{k_{p-1}}{\lambda}) S_\lambda^{[p-1]}(h,\frac{k_{p-1}}{\lambda}),$$

where $Q_m(\frac{k_{p-1}}{\lambda})$ is a polynomial in $\frac{k_{p-1}}{\lambda}$ of degree m with coefficients as polynomials in $\frac{1}{\lambda}$ of degree not exceeding m.

Now for $o \leq m_o \leq m$ we have

$$(5.2.22) \frac{1}{\lambda+1} \sum_{k_{p-1}=o}^{\lambda} (\frac{k_{p-1}}{\lambda})^{m_o} S_\lambda^{[p-1]} (h, \frac{k_{p-1}}{\lambda}) = (\frac{\lambda}{\lambda+1}) \cdot$$

$$\cdot (\int_o^1 t^{m_o} S_\lambda^{[p-1]} (h,t)\, dt + R).$$

Here, when $m_o = o$, as previously (via the Euler-Maclaurin sum formula)

$$R = O(\lambda^{-(k+1)}) \; ||h||_{C^{2k+1}}.$$

Also, when $1 \leq m_o \leq m$, for $k = o$,

$$R = \frac{1}{2\lambda} [S_\lambda^{[p-1]} (h,1) + T] ,$$

where $|T| \leq m_o \; || S_\lambda^{[p-1]} h|| + || S_\lambda^{[p-1](1)} h||$; and for $k \neq o$,

$$R = \frac{1}{2\lambda} S_\lambda^{[p-1]}(h,1) + \frac{B_2}{2!\,\lambda^2} S_\lambda^{[p-1](1)}(h,1)$$

$$+ \frac{B_4}{4!\,\lambda^4} S_\lambda^{[p-1](3)}(h,1) + \dots + \frac{B_{2k-2}}{(2k-2)!\,\lambda^{2k-2}} S_\lambda^{[p-1](2k-3)}(h,$$

$$- \frac{1}{2k!\,\lambda^{2k+1}} \sum_{m'=o}^{\lambda-1} \int_o^1 \beta_{2k}(t) [\frac{d^{2k}}{dt^{2k}} (t^{m_o} S_\lambda^{[p-1]} (h,t))]_{t=\lambda^{-1}(t+}$$

$$\cdot dt.$$

Hence, if $k = o$ in (5.2.22), using (5.2.11) and (5.2.14-15) we have

$$|R| = O(1/\lambda) \; ||h||_{C'} .$$

Also, for $k \neq o$, using (5.2.11) and (5.2.14-15) again it is easy to see that

$$|R| = O(1/\lambda^{k+1}) \ ||h||_{C^{2k+1}}.$$

Thus in both the cases ($k = o$ and $k \neq o$)

$$|R| = O(\lambda^{-(k+1)}) \ ||h||_{C^{2k+1}} .$$

Hence for all $o \leq m_o \leq m$ we have

$$\frac{1}{\lambda+1} \sum_{k_{p-1}=o}^{\lambda} \left(\frac{k_{p-1}}{\lambda}\right)^{m_o} S_\lambda^{[p-1]} \left(h, \frac{k_{p-1}}{\lambda}\right)$$

$$= \frac{\lambda}{\lambda+1} \Big[C_o(p-1,m_o,h) + \frac{C_1(p-1,m_o,h)}{\lambda} + \dots + \frac{C_k(p-1,m_o,h)}{\lambda^k}$$

$$+ O(\lambda^{-(k+1)}) \ ||h||_{C^{2k+1}} \Big].$$

Thus, by (5.2.21) we can write

$$\int_a^b h(u) \int_o^1 W^{[p]}(\lambda, t, u) \ t^m \, dt \, du$$

$$= C_o(p,m,h) + \frac{C_1(p,m,h)}{\lambda} + \dots + \frac{C_k(p,m,h)}{\lambda^k}$$

$$+ O(\lambda^{-(k+1)}) \ ||h||_{C^{2k+1}},$$

where $C_i(p,m,h)$'s are certain combinations of $C_j(p-1,m_o,h)$'s, showing that (5.2.8) is true for p and $m \in \mathbb{N}$. But, since (5.2.8) holds for $p = 1$ and $m \in \mathbb{N}^o$ and also for $m = o$ and all $p \in \mathbb{N}$, it follows that it is valid for all $p \in \mathbb{N}$ and $m \in \mathbb{N}^o$.

Now, since $h \in C_o^{2k+1}$ with supp $h \subset (a,b)$, we have

$$\int_a^b h(u) \int_0^1 W^{[p]} (\lambda,t,u)\, t^m \, dt\, du$$

$$= \int_0^1 t^m S_\lambda^{[p]} (h,t)\, dt$$

$$= \int_0^1 t^m \Big[\sum_{j=0}^{2k+1} \frac{h^{(j)}(t)}{j!} \mu_{\lambda,j}^{[p]}(t) + o(\lambda^{-(2k+1)/2}) \|h\|_{C^{2k+1}} \Big] dt$$

(5.2.23)

$$= \sum_{j=0}^{2k+1} \frac{(-1)^j}{j!} \int_0^1 h(t) \{ t^m \mu_{\lambda,j}^{[p]} (t) \}^{(j)}\, dt$$

$$+ o(\lambda^{-(2k+1)/2}) \|h\|_{C^{2k+1}}$$

$$= \int_0^1 h(t) \Big\{ \sum_{j=0}^{2k+1} \frac{(-1)^j}{j!} [t^m \mu_{\lambda,j}^{[p]} (t)]^{(j)} \Big\} dt +$$

$$+ o(\lambda^{-(2k+1)/2}) \|h\|_{C^{2k+1}} .$$

A comparison of (5.2.23) and (5.2.8) leads us to

$$\int_a^b h(u) \int_0^1 W^{[p]} (\lambda,t,u)\, t^m \, dt\, du$$

$$= \int_0^1 h(t) \Big\{ \sum_{j=0}^{2k+1} \frac{(-1)^j}{j!} [t^m \mu_{\lambda,j}^{[p]} (t)]^{(j)} \Big\} dt$$

$$+ O(\lambda^{-(k+1)}) \|h\|_{C^{2k+1}} ,$$

which completes the proof of Lemma 5.2.4.

Continuing with the proof of Lemma 5.2.3 we now estimate I_2.

$$|I_2| \le \int_0^1 \int_{\text{supp } f} \sum_{j=0}^{k} |C(j,k)| \; W^{[p]} (d_j \lambda_{n_r}, t, u) |f(u)| \; \cdot$$

$$\cdot \; |\varepsilon(t,u)| \, (u-t)^{2k+2} \, du\, dt.$$

Since, for some ξ lying between t and u,

$$|\epsilon(t,u)| = \frac{|g^{(2k+2)}(\xi) - g^{(2k+2)}(u)|}{(2k+2)!} \leq \frac{2}{(2k+2)!} \; ||g^{(2k+2)}||_{C[a,}$$

and

$$|f(u)| \leq ||f||_{C^{2k+1}} \cdot$$

It follows that

$$|I_2| \leq M_3 \; ||f||_{C^{2k+1}} \sum_{j=o}^{k} |C(j,k)| \cdot \underset{o \leq j \leq k}{\text{Max}} \int_o^1 \int_o^1 W^{[p]}(d_j \lambda_{n_r}, t,u) \cdot$$
$$\cdot (u-t)^{2k+2} \, du \, dt$$

$$= O(\lambda_{n_r}^{-(k+1)}) \; ||f||_{C^{2k+1}} \cdot$$

Then, using Proposition 5.2.2 and (5.2.8) it is clear that

$$\lambda_{n_r}^{k+1} < S_{\lambda_{n_{r+1}},p}(f,k,t) - S_{\lambda_{n_r},p}(f,k,t), g(t) >$$

$$= \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \sum_{j=o}^{k} C(j,k) \int_{\text{supp}\, f} \phi_i(u) \int_o^1 W^{[p]}(d_j \lambda_{n_r}, t,u) t^i \, dt \, du$$

$$- \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \sum_{j=o}^{k} C(j,k) \int_{\text{supp}\, f} \phi_i(u) \int_o^1 W^{[p]}(d_j \lambda_{n_{r+1}}, t,u) \cdot$$
$$\cdot t^i \, dt \, du + O(1) \; ||f||_{C^{2k+1}}$$

$$= O(1) \; ||f||_{C^{2k+1}} \cdot$$

This completes the proof of Lemma 5.2.3.

Continuing the proof of Theorem 5.2.1, since $C^{2k+2}[o,1]$ is dense in $C^{2k+1}[o,1]$ with respect to $||\cdot||_{C^{2k+1}}$, there exists a sequence $\{f_\sigma^*\}$ in $C^{2k+2}[o,1]$, converging to f^* in the $||\cdot||_{C^{2k+1}}$ norm.

Therefore by Lemma 5.2.3 we get

$$|\lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}}},p(f_\sigma^*-f^*,k,t) - S_{\lambda_{n_{r_q}}},p(f_\sigma^*-f^*,k,t),g(t) > |$$

$$\leq M \; ||f_\sigma^* - f^*||_{C^{2k+1}} \; .$$

Consequently,

$$\lim_{\sigma\to\infty} \lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q}+1}},p(f_\sigma^* - f^*,k,t) - S_{\lambda_{n_{r_q}}},p(f_\sigma^* - f^*,k,t),$$

$$g(t) > = o,$$

which can be equivalently written as

$$\lim_{\sigma\to\infty} \lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q}+1}},p(f_\sigma^*,k,t) - S_{\lambda_{n_{r_q}}},p(f_\sigma^*,k,t),g(t)>$$

(5.2.24)

$$= \lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q}+1}},p(f^*,k,t) - S_{\lambda_{n_{r_q}}},p(f^*,k,t),g(t) > \; .$$

Now, since $f_\sigma^* \in C^{2k+2}[o,1]$, by Theorem 4.2.2 and Proposition 5.2.2 we get

$$\lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}}},p(f_\sigma^*,k,t) - S_{\lambda_{n_{r_q}}},p(f_\sigma^*,k,t),g(t) >$$

$$= < -(1-(\tfrac{1}{\beta})^{k+1}) \sum_{j=o}^{2k+2} Q(j,k,p,t)\, f_\sigma^{*(j)}(t),g(t) >$$

(5.2.25)

$$= < P_{2k+2}(D)\, f_\sigma^*,g(t)>$$

$$= < f_\sigma^*,P_{2k+2}^*(D)g >,$$

where $P^*_{2k+2}(D)$ is the dual operator of $P_{2k+2}(D)$.
Thus combining (5.2.25) and (5.2.24-25) we have

$$\lim_{\sigma \to \infty} \langle f^*_\sigma, P^*_{2k+2}(D)g \rangle = \langle h(t), g(t) \rangle,$$

i.e.,

$$\langle f^*, P^*_{2k+2}(D)g \rangle = \langle h(t), g(t) \rangle,$$

for all $g \in C_0^\infty$ with supp $g \subset (a_2^*, b_2^*)$.
Thus, treating f^* as a generalized function,

(5.2.26) $P_{2k+2}(D)\, f^*(t) = h(t)$.

Now, since $Q(2k+2,k,p,t) \neq o$ (by the remark after Theorem 4.2.2), interpreting (5.2.26) as a first order linear differential equation for $f^{*(2k+1)}$ we deduce that $f^{*(2k+1)} \in$ A.C.$[a_2^*, b_2^*]$ and that $f^{*(2k+2)} \in L_\infty[a_2^*, b_2^*]$. Noticing that $f^* \equiv f$ on $[a_2, b_2]$ and $[a_2, b_2] \subset [a_2^*, b_2^*]$, (ii) is immediate. This completes the proof of the implication (i) $\Rightarrow$(ii).

That (ii) $\Longrightarrow$(iii) follows immediately by Theorem 4.2.3.

To prove (iv) $\Rightarrow$ (v), assuming (iv) and proceeding as in the proof of (i) $\Rightarrow$ (ii) we get

$$P_{2k+2}(D)\, f^* = o,$$

from which (v) follows.

Lastly, (v) $\Rightarrow$ (vi) is clear by Theorem 4.2.2.

This completes the proof of Theorem 5.2.1.

5.3 SATURATION THEOREM FOR $S_{\lambda,k+1,m}(\cdot,t)$

The generalized Micchelli type combinations of the Bernstein polynomials are governed by the following saturation

THEOREM 5.3.1 : If $k,m \in \mathbb{N}^{o}$, in the following statements the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are true :

(i) $\lambda_n^{k+1} \sup\limits_{t\in[a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = O(1)$;

(ii) $f^{(2k+1)} \in A.C.\,[a_2,b_2]$ and $f^{(2k+2)} \in L_\infty[a_2,b_2]$;

(iii) $\lambda^{k+1} \quad ||S_{\lambda,k+1,m}(f,t) - f(t)||_{C[a_3,b_3]} = O(1)$;

(iv) $\lambda_n^{k+1} \sup\limits_{t\in[a_1,b_1]} |S_{\lambda_n,k+1,m}(f,t)-f(t)| = o(1)$;

(v) $f \in C^{2k+2}[a_2,b_2]$ and $\sum\limits_{j=2}^{2k+2} Q(j,k+1,m,t)f^{(j)}(t) = o, t \in [a_2$

where $Q(j,k+1,m,t)$ are the polynomials occuring in (2.3.1)

(vi) $\lambda^{k+1} \quad ||S_{\lambda,k+1,m}(f,t)-f(t)||_{C[a_3,b_3]} = o(1)$.

PROOF : First assume (i). Then, as in the proof of (i) $\Rightarrow$ (ii) of Theorem 2.5.1 it follows that f is continuous on $[a_1,b_1]$ and that $f^{(2k+1)}$ exists and is continous on (a_1,b_1). Let a_1^*,a_2^*,b_1^*,b_2^* and f^* be as in the proof of Theorem 2.5.1. Proceeding as in the proof of (i) $\Rightarrow$ (ii) of Theorem 5.2.1 we find that there exists an $h \in L_\infty[a_2^*,b_2^*]$ and a sequence $\{r_q \in \mathbb{N}\}_{q=1}^{\infty}$ such that

$$\lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q}+1},k+1,m}(f^*,t) - S_{\lambda_{n_{r_q}},k+1,m}(f^*,t), g(t) >$$

$$\to < h(t), g(t) >, \tag{5.3.1}$$

for any $g \in C_o^\infty(a_2^*, b_2^*)$.

Now, as an intermediate result we prove the analogue of Le 5.2.3 in the present case.

LEMMA 5.3.2 : Let $g \in C_o^\infty(a_2^*, b_2^*)$ and $f \in C_o^{2k+1}(a_1, b_1)$. Then there holds the inequality

$$| \lambda_{n_r}^{k+1} < S_{\lambda_{n_{r+1}},k+1,m}(f,t) - S_{\lambda_{n_r},k+1,m}(f,t), g(t) > | \tag{5.3.2}$$

$$\leq M \; ||f||_{C^{2k+1}},$$

for all sufficiently large r, where M does not depend on f and

PROOF : Proceeding as in the proof of Lemma 5.2.3 we get

$$< S_{\lambda_{n_r},k+1,m}(f,t), g(t) >$$

$$= \sum_{i=o}^{2k+2} \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} \int_{\mathrm{supp}\, f} \int_o^1 \cdot$$

$$\cdot\; W^{[p]}(\lambda_{n_r}, t, u)\, \phi_i(u)\, t^i \, dt\, du$$

$$+ \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} \int_o^1 \cdot \tag{5.3.3}$$

$$\cdot \int_{\mathrm{supp}\, f} W^{[p]}(\lambda_{n_r}, t, u)\, f(u)\, \varepsilon(t,u)(t-u)^{2k+2}\, du\, dt$$

$$= I_1 + I_2, \text{ say.}$$

Then, as previously,

(5.3.4) $$|I_2| = O(\lambda_{n_r}^{-(k+1)})\ ||f||_{C^{2k+1}} .$$

Hence, using Proposition 5.2.2, (5.3.3-4), Lemma 5.3.4 and Lemma 2.2.4 we get

$$\lambda_{n_r}^{k+1} < S_{\lambda_{n_{r+1}},k+1,m}(f,t) - S_{\lambda_{n_r},k+1,m}(f,t), g(t) >$$

$$= \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \sum_{p=m+1}^{2k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!\ (p-m-1)!} \binom{k+m+1}{p} \cdot$$

$$\cdot \int_{\text{supp } f} \int_o^1 W^{[p]}(\lambda_{n_r}, t, u)\ \phi_i(u)\ t^i\ dt\ du$$

$$- \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}\ (p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} \cdot$$

$$\cdot \int_{\text{supp } f} \int_o^1 W^{[p]}(\lambda_{n_{r+1}}, t, u)\ \phi_i(u)\ t^i\ dt\ du$$

$$+ O(1)\ ||f||_{C^{2k+1}}$$

$$= \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \int_o^1 \phi_i(t) [\sum_{j=o}^{2k+1} \frac{(-1)^j}{j!} \cdot$$

$$\cdot \{ t^i\ S_{\lambda_{n_r},k+1,m}((u-t)^j, t) \}^{(j)}]\ dt$$

$$- \lambda_{n_r}^{k+1} \sum_{i=o}^{2k+2} \int_o^1 \phi_i(t) [\sum_{j=o}^{2k+1} \frac{(-1)^j}{j!} \cdot$$

$$\cdot \{ t^i\ S_{\lambda_{n_{r+1}},k+1,m}((u-t)^j, t) \}^{(j)}]\ dt + O(1)\ ||f||_{C^{2k+1}}$$

$$= O(1)\ ||f||_{C^{2k+1}} .$$

Let f_σ^* be as in the proof of Theorem 5.2.1. Then, by Lemma 5.3.2

$$\left| \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}},k+1,m}(f_\sigma^*-f^*,t) - S_{\lambda_{n_{r_q}},k+1,m}(f_\sigma^*-f^*,t), g(t) > \right| \leq M \, ||f_\sigma^*-f^*||_{C^{2k+1}} .$$

Hence,

$$\lim_{\sigma \to \infty} \lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}},k+1,m}(f_\sigma^*,t) - S_{\lambda_{n_{r_q}},k+1,m}(f_\sigma^*,t), g(t) >$$

(5.3.5)

$$= \lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}},k+1,m}(f^*,t) - S_{\lambda_{n_{r_q}},k+1,m}(f^*,t), g(t) > .$$

Now, since $f_\sigma^* \in C^{2k+2}[o,1]$, by Theorem 2.3.1 and Proposition 5.2.2 we have

$$\lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S_{\lambda_{n_{r_q+1}},k+1,m}(f_\sigma^*,t) - S_{\lambda_{n_{r_q}},k+1,m}(f_\sigma^*,t), g(t) >$$

$$= < Q_{2k+2}(D) \, f_\sigma^*, g(t) >$$

(5.3.6)

$$= < f_\sigma^*, Q_{2k+2}^*(D) g > ,$$

where $Q_{2k+2}(D) = -(1 - (\frac{1}{\beta})^{k+1}) \sum_{j=o}^{2k+2} Q(j,k+1,m,t) D^j$, $Q_{2k+2}^*(D)$ is the dual operator of $Q_{2k+2}(D)$ and $Q(j,k+1,m,t)$'s are as in the statement of Theorem 5.3.1.

Thus combining (5.3.1) and (5.3.5-6) we get

$$\langle f^*, Q^*_{2k+2}(D)g\rangle = \langle h(t), g(t)\rangle ,$$

for all $g \in C_0^\infty$ with supp $g \subset (a_2^*, b_2^*)$.

As noted in the proof of Theorem 2.5.1, $Q(2k+2,k+1,m,t) \neq$ (in view of which (ii) follows.

The statement (ii) $\Rightarrow$ (iii) follows from Theorem 2.3.2. A formal proof of (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) follows from the above proof along similar lines as in that of Theorem 2.5.1 and is omitted.

This completes the proof of Theorem 5.3.1.

COROLLARY 5.3.3 : If k and $m \in \mathbb{N}^o$, then for each $i = 1,2,\ldots,k$ there holds the identity

$$(5.3.7) \quad \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} \binom{p}{i} = 0.$$

PROOF : The function $f(u) = u(1-u)$ satisfies the hypothesis of the statement (ii) of Theorem 5.3.1. Hence, by (ii) $\Rightarrow$ (iii) of the same theorem for any $t \in (0,1)$ we have

$$(5.3.8) \quad S_{\lambda,k+1,m}(u(1-u),t) - t(1-t) = O(\lambda^{-(k+1)}).$$

Also,

$$S_{\lambda,k+1,m}(u(1-u),t)$$

$$= \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} S_\lambda^p(u(1-u),t)$$

(5.3.9)

$$= \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} t(1-t)\,(1-\tfrac{1}{\lambda})^p$$

$$= \sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!\,(p-m-1)!} \binom{k+m+1}{p}\; t(1-t)\left[\sum_{i=0}^{p} \binom{p}{i} \left(-\frac{1}{\lambda}\right)^i\right].$$

Combining (5.3.8) and (5.3.9) it follows that

$$\sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p}\; t(1-t)\left[\sum_{i=1}^{p} \binom{p}{i} \left(-\frac{1}{\lambda}\right)^i\right]$$

$$= O\left(\lambda^{-(k+1)}\right).$$

It is possible only when

$$\sum_{p=m+1}^{k+m+1} (-1)^m \frac{(-1)^{p+1}(p-1)!}{m!(p-m-1)!} \binom{k+m+1}{p} \binom{p}{i} = 0,$$

for each $i = 1,2,\ldots,k$, completing the proof.

Note that (5.3.7) is equivalent to the identity (2.5.4).

5.4 SATURATION THEOREM FOR $S^{(m)}_{\lambda,p}(\cdot,k,t)$

In this section we prove the analogue of Theorem 5.2.1 in simultaneous approximation.

THEOREM 5.4.1 : If $k,m, \in \mathrm{IN}^{o}$ and $p \in \mathrm{IN}$, in the following statements the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) hold :

(i) $f^{(m)}$ exists on $[a_1,b_1]$ and

$$\lambda_n^{k+1} \sup_{t\in[a_1,b_1]} \left| S^{(m)}_{\lambda_n,p}(f,k,t) - f^{(m)}(t)\right| = O(1) ;$$

(ii) $f^{(2k+m+1)} \in$ A.C. $[a_2,b_2]$ and $f^{(2k+m+2)} \in L_\infty[a_2,b_2]$;

(iii) $\lambda^{k+1} \left\| S^{(m)}_{\lambda,p}(f,k,t) - f^{(m)}(t)\right\|_{C[a_3,b_3]} = O(1)$;

(iv) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1, b_1]} |S^{(m)}_{\lambda_n, p}(f,k,t) - f^{(m)}(t)| = o(1) \;;$$

(v) $f^{(2k+m+2)} \in C[a_2, b_2]$ and $\sum_{j=m}^{2k+m+2} Q(j,k,m,p,t)\, f^{(j)}(t) = 0$, $t \in [a_2, b_2]$ where $Q(j,k,m,p,t)$ are the polynomials occuring in (4.2.2) ;

(vi) $$\lambda^{k+1} \,||S^{(m)}_{\lambda,p}(f,k,t) - f^{(m)}(t)||_{C[a_3,b_3]} = o(1).$$

PROOF : First let us assume (i). Then as in the proof of (i) $\Rightarrow$ (ii) of Theorem 4.4.1, $f^{(m)} \in C[a_1, b_1]$ and $f^{(2k+m+1)}$ exists and is continuous on (a_1, b_1). Let $a_1^*, a_2^*, b_1^*, b_2^*$ and f^* be as in the proof of (i) $\Rightarrow$ (ii) of Theorem 4.4.1. Then, by Theorem 4.2.2. we have

$$(5.4.1)\quad ||S^{(m)}_{\lambda_n,p}(f^*,k,t) - f^{*(m)}(t)||_{C[a_2^*, b_2^*]} = O(\lambda_n^{-(k+1)}).$$

In particular, for $r \in \mathbb{N}$ (in the context of Proposition 5.2.2)

$$(5.4.2)\quad ||S^{(m)}_{\lambda_{n_r},p}(f^*,k,t) - f^{*(m)}(t)||_{C[a_2^*, b_2^*]} = O(\lambda_{n_r}^{-(k+1)}).$$

Then, following the proof of (i) $\Rightarrow$ (ii) of Theorem 5.2.1 it is easy to see that (5.4.2) is equivalent to

$$(5.4.3)\quad \begin{aligned} &||S^{(m)}_{\lambda_{n_{r+1}},p}(f^*,k,t) - S^{(m)}_{\lambda_{n_r},p}(f^*,k,t)||_{C[a_2^*, b_2^*]} \\ &\qquad = O(\lambda_{n_r}^{-(k+1)}), \end{aligned}$$

and that (5.4.3) yields

$$\lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S^{(m)}_{\lambda_{n_{r_q}+1}, p}(f^*,k,t) - S^{(m)}_{\lambda_{n_{r_q}}, p}(f^*,k,t), g(t) >$$

(5.4.4)

$$= < h(t), g(t) > ,$$

for every $g \in C_0^{\infty}(a_2^*, b_2^*)$, some $h \in L_{\infty}[a_2^*, b_2^*]$ and some sequence $\{ r_q \}_{q=1}^{\infty}$ of natural numbers.

Now we shall prove the following lemma :

<u>LEMMA</u> 5.4.2 : Let $g \in C_0^{\infty}(a_2^*, b_2^*)$ and $f \in C_0^{2k+m+1}(a_1, b_1)$, then

$$\lambda_{n_r}^{k+1} \left| < S^{(m)}_{\lambda_{n_{r+1}}, p}(f,k,t) - S^{(m)}_{\lambda_{n_r}, p}(f,k,t), g(t) > \right|$$

$$\leq M \, ||f||_{C^{2k+m+1}} ,$$

for all r sufficiently large, where M does not depend on f and r.

<u>PROOF</u> : Since $g \in C_0^{\infty}(a_2^*, b_2^*)$, an application of Lemma 5.2.3 yields

$$\lambda_{n_r}^{k+1} \left| < S^{(m)}_{\lambda_{n_{r+1}}, p}(f,k,t) - S^{(m)}_{\lambda_{n_r}, p}(f,k,t), g(t) > \right|$$

$$= \lambda_{n_r}^{k+1} \left| < S_{\lambda_{n_{r+1}}, p}(f,k,t) - S_{\lambda_{n_r}, p}(f,k,t), g^{(m)}(t) > \right|$$

$$\leq M_1 \, ||f||_{C^{2k+1}} \leq M \, ||f||_{C^{2k+m+1}},$$

where M_1 and M do not depend on f or r.

This completes the proof of the lemma.

Now, since $C^{2k+m+2}[o,1]$ is dense in $C^{2k+m+1}[o,1]$ with respect to $||\cdot||_{C^{2k+m+1}}$, there exists a sequence $\{f_\sigma^*\}$ in $C^{2k+m+2}[o,1]$ such that $||f_\sigma^*-f^*||_{C^{2k+m+1}} \to o$ as $\sigma \to \infty$.

Hence, by Lemma 5.4.2 we get

$$\lim_{\sigma\to\infty} \lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S^{(m)}_{\lambda_{n_{r_q}+1}, p}(f_\sigma^*,k,t) - S^{(m)}_{\lambda_{n_{r_q}}, p}(f_\sigma^*,k,t), g(t) >$$

(5.4.5)

$$= \lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S^{(m)}_{\lambda_{n_{r_q}+1}, p}(f^*,k,t) - S^{(m)}_{\lambda_{n_{r_q}}, p}(f^*,k,t), g(t) > .$$

Now, since $f_\sigma^* \in C^{2k+m+2}[o,1]$, by Theorem 3.2.2 and Proposition 5.2.2 we have

$$\lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S^{(m)}_{\lambda_{n_{r_q}+1}, p}(f_\sigma^*,k,t) - S^{(m)}_{\lambda_{n_{r_q}}, p}(f_\sigma^*,k,t), g(t) >$$

(5.4.6)

$$= < -(1-(\tfrac{1}{\beta})^{k+1}) \sum_{j=m}^{2k+m+2} Q(j,k,m,p,t) f_\sigma^{*(j)}(t), g(t) >$$

$$= < f_\sigma^*(t), -(1-(\tfrac{1}{\beta})^{k+1}) \sum_{j=1}^{2k+m+2} Q^*(j,k,m,p,t) g^{(j)}(t) >,$$

where $P^*_{2k+m+2}(D) = -(1-(\frac{1}{\beta})^{k+1}) \sum_{j=1}^{2k+m+2} Q^*(j,k,m,p,t)\, D^j$ denotes the operator adjoint to $P_{2k+m+2}(D) = -(1-(\frac{1}{\beta})^{k+1}) \sum_{j=m}^{2k+m+2} Q(j,k,m,p,t)\, D^j$.

Thus combining (5.4.4-6) we get

(5.4.7) $\quad < f^*, P^*_{2k+m+2}(D)g > \; = \; < h(t), g(t) >,$

for all $g \in C_o^\infty(a_2^*, b_2^*)$.

Now, since $Q(2k+m+2,k,m,p,t) \neq o$ by the remark after theorem 4.2.2,(ii) follows. This completes the proof of the implication (i) $\Rightarrow$ (ii).

The assertion (ii) $\Rightarrow$ (iii) follows from Theorem 4.2.3.

For (iv) $\Rightarrow$ (v), assuming (iv) and proceeding as in the proof of (i) $\Rightarrow$ (ii) we have

$$P_{2k+m+2}(D)\, f^* = o,$$

from which (v) is clear.

Lastly, (v) $\Rightarrow$ (vi) follows from Theorem 4.2.2.

This completes the proof of Theorem 5.4.1.

5.5. SATURATION THEOREM FOR $S^{(p)}_{\lambda,k+1,m}(\cdot,t)$

In our last result we establish the analogue of Theorem 5.3.1 in simultaneous approximation.

THEOREM 5.5.1 : In the following statements, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are valid for any $k, m \in \mathbb{N}^o$ and $p \in \mathbb{N}$.

(i) $f^{(p)}$ exists on $[a_1,b_1]$ and

$$\lambda_n^{k+1} \sup_{t\in[a_1,b_1]} |S^{(p)}_{\lambda_n,k+1,m}(f,t)-f^{(p)}(t)| = O(1) ;$$

(ii) $f^{(2k+p+1)} \in A.C.\,[a_2,b_2]$ and $f^{(2k+p+2)} \in L_\infty[a_2,b_2]$;

(iii) $\lambda^{k+1}\, ||\, S^{(p)}_{\lambda,k+1,m}(f,t)-f^{(p)}(t)\, ||_{C[a_3,b_3]} = O(1)$;

(iv) $f^{(p)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1,b_1]} | S^{(p)}_{\lambda_n,k+1,m}(f,t) - f^{(p)}(t) | = o(1) \;;$$

(v) $f \in C^{2k+p+2}[a_2,b_2]$ and $\sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t)\, f^{(j)}(t) = o,$

$t \in [a_2,b_2]$, where $Q(j,k+1,m,p,t)$ are the polynomials occuring in (3.2.2) ;

(vi) $\lambda^{k+1} \| S^{(p)}_{\lambda,k+1,m}(f,t) - f^{(p)}(t) \|_{C[a_3,b_3]} = o(1).$

<u>PROOF</u> : If (i) holds, following the proof of (i) $\Rightarrow$ (ii) of Theorem 3.4.1 it follows that $f^{(p)}$ is continuous on $[a_1,b_1]$ and that $f^{(2k+p+1)}$ exists and is continuous on (a_1,b_1). Let $a_1^*, a_2^*, b_1^*, b_2^*$ and f^* be as in the proof of (i) $\Rightarrow$ (ii) of Theorem 3.4.1. Then, by Theorem 3.2.2 we have

$$(5.5.1) \quad \| S^{(p)}_{\lambda_n,k+1,m}(f^*,t) - f^{*(p)}(t) \|_{C[a_2^*,b_2^*]} = O(\lambda_n^{-(k+1)}).$$

Proceeding as in the proof of (i) $\Rightarrow$(ii) of Theorem 5.4.1 we find that for some sequence $\{ r_q \in \mathbb{N} \}_{q=1}^{\infty}$

$$\lim_{q\to\infty} \lambda_{n_{r_q}}^{k+1} < S^{(p)}_{\lambda_{n_{r_q+1}},k+1,m}(f^*,t) - S^{(p)}_{\lambda_{n_{r_q}},k+1,m}(f^*,t), g(t) >$$

$$(5.5.2) \qquad = < h(t), g(t) > ,$$

for every $g \in C_0^{\infty}(a_2^*, b_2^*)$, where $h \in L_\infty[a_2^*, b_2^*]$ is a fixed function.

Next, for any $g \in C_0^{\infty}(a_2^*, b_2^*)$ and $s \in C_0^{2k+p+1}(a_1, b_1)$, following the proof of Lemma 5.4.2 and making an application of Lemma 5.3.2 we obtain

$$\lambda_{n_r}^{k+1} \left| < S^{(p)}_{\lambda_{n_{r+1}},k+1,m}(s,t) - S^{(p)}_{\lambda_{n_r},k+1,m}(s,t), g(t) > \right|$$

$$(5.5.3) \qquad \leq M \, ||s||_{C^{2k+p+1}} ,$$

for all r sufficiently large, where M is independent of s or r.

Now, as in the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 5.4.1, there exists a sequence $\{f_\sigma^*\}$ in $C^{2k+p+2}[0,1]$ converging to f^* in $||\cdot||_{C^{2k+p+1}}$ norm.

Hence, in particular from (5.5.3) we have

$$\lim_{\sigma \to \infty} \lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S^{(p)}_{\lambda_{n_{r_q+1}},k+1,m}(f_\sigma^*,t) - S^{(p)}_{\lambda_{n_{r_q}},k+1,m}(f_\sigma^*,t), g(t) >$$

(5.5.4)

$$= \lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S^{(p)}_{\lambda_{n_{r_q+1}},k+1,m}(f^*,t) - S^{(p)}_{\lambda_{n_{r_q}},k+1,m}(f^*,t), g(t) >$$

Also, since $f_\sigma^* \in C^{2k+p+2}[0,1]$, by Proposition 5.2.2 and Theorem 3.2.2 we get

$$\lim_{q \to \infty} \lambda_{n_{r_q}}^{k+1} < S^{(p)}_{\lambda_{n_{r_q+1}},k+1,m}(f_\sigma^*,t) - S^{(p)}_{\lambda_{n_{r_q}},k+1,m}(f_\sigma^*,t), g(t) >$$

$$(5.5.5) \quad = \left\langle -\left(1-\left(\tfrac{1}{\beta}\right)^{k+1}\right) \sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t) f_\sigma^{*(j)}(t), g(t) \right\rangle$$

$$= \left\langle f_\sigma^{*}(t), -\left(1-\left(\tfrac{1}{\beta}\right)^{k+1}\right) \sum_{j=1}^{2k+p+2} Q^{*}(j,k+1,m,p,t) g^{(j)}(t) \right\rangle ,$$

where as in the previous theorem $Q^{*}_{2k+p+2}(D) = -(1-(\frac{1}{\beta})^{k+1}) \sum_{j=1}^{2k+p+2} Q^{*}(j,k+1,m,p,t)\, D^{j}$ denotes the operator adjoint to $Q_{2k+p+2}(D) = -(1-(\frac{1}{\beta})^{k+1}) \sum_{j=p}^{2k+p+2} Q(j,k+1,m,p,t)\, D^{j}$.

Thus, by (5.5.2) and (5.5.4-5) we have

$$\langle f^{*}, Q^{*}_{2k+p+2}(D) g \rangle = \langle h(t), g(t) \rangle .$$

Now, since $Q(2k+p+2,k+1,m,p,t) > 0$ from the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 3.4.1, (ii) follows.

That (ii) $\Rightarrow$(iii) is immediate from Theorem 3.2.3.

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) follow from the above proof along similar lines as in that of Theorem 4.4.1.

This establishes our last result.

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